EXACT WIRELENGTH OF EMBEDDING THE HYPERCUBES INTO CYCLE-OF-LADDERS

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Abstract

Hypercubes are a very popular model for parallel computation because of their regularity and the relatively small number of interprocessor connections. In this paper, we present an algorithm to compute the exact wirelength of embedding the hypercubes into cycle-of-ladders and prove its correctness.

Keywords - Embedding, dilation, wirelength, hypercubes, cycle-of-ladders.

1. INTRODUCTION

Graph embedding is an important technique that maps a logical graph into a host graph, usually an interconnection network. Many applications can be modeled as graph embedding. For example architecture simulation, which is to simulate one architecture by another, can be modeled as embedding the guest architecture into the host architecture. Another example is the process allocation. In parallel computing, a large process is often decomposed into a set of small sub-processes that can execute in parallel with communications among these sub-processes. The problem of allocating these sub-processes into a parallel computing system can be again modeled by graph embedding [1].

In recent years, among many interconnection networks, the hypercube has been the focus of many researchers due to its structural regularity, potential for parallel computation of various algorithms, and the high degree of fault tolerance [2]. Hypercubes are known to simulate other structures such as grids and binary trees [3, 4].

The quality of an embedding can be measured by certain cost criteria. One of these criteria which is considered very often is the dilation. The dilation of an embedding is defined as the maximum distance between a pair of vertices of host graph that are images of adjacent vertices of logical graph. It is a measure for the communication time needed when simulation one network on another [5]. Another important cost criteria is the wirelength. The wirelength of a graph embedding arises from VLSI designs, data structures and data representations, networks for parallel computer systems, biological models that deal with cloning and visual stimuli, parallel architecture, structural engineering and so on [6, 7].

Graph embeddings have been well studied for binary trees into paths [7], binary trees into hypercubes [4, 5], complete binary trees into hypercubes [8], meshes into crossed cubes [9], meshes into locally twisted cubes [10], meshes into faulty crossed cubes [11], generalized ladders into hypercubes [12], grids into grids [13], hypercubes into cycles [14, 15], star graph into path [16], snarks into torus [17], generalized wheels into arbitrary trees [18], hypercubes into grids [3], m-sequencial k-ary trees into hypercubes [19], meshes into m¨obius cubes [20], ternary tree into hypercube [21], enhanced and augmented hypercube into complete binary tree [22], circulant into arbitrary trees, cycles, certain multicyclic graphs and ladders [23], hypercubes into cylinders, snakes and caterpillars [24], tori and grids into twisted cubes [25], incomplete hypercube in books [26], hypercubes into necklace, windmill and snake graphs [27], embedding of special classes of circulant networks, hypercubes and generalized Petersen graphs [28], embedding variants of hypercubes with dilation 2 [29].

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Even though there are numerous results and discussions on the wirelength problem, most of them deal with only approximate results and the estimation of lower bounds [17]. The embeddings discussed in this paper produce exact wirelength.

2. BASIC CONCEPTS

***Definition* 2.1.**[8] Let G and H be finite graphs with n vertices. $V(G)$ and $V(H)$ denote the vertex sets of G and H respectively. $E(G)$ and $E(H)$ denote the edge sets of G and H respectively. An *embedding* f of G into H is defined as follows:
(i) f is a injective map from $V(G) \to V(H)$
(ii) P_f is an injective map from $E(G)$ to $\{P_f(f(u), f(v)) : P_f(f(u), f(v))$ is a path in H between $f(u)$ and $f(v)\}$.
The graph G that is being embedded is called a *virtual graph* or a *guest graph* and H is called a *host graph*. Some authors use the name *labeling* instead of embedding.

The *edge congestion* of an embedding f of G into H is the maximum number of edges of the graph G that are embedded on any single edge of H. Let $EC_f(G, H(e))$ denote the number of edges (u, v) of G such that e is in the path $P_f(u, v)$ between $f(u)$ and $f(v)$ in H. In other words,

$$EC_f(G, H(e)) = \left|\{(u,v) \in E(G) : e \in P_f(u,v)\}\right|$$

where $P_f(u, v)$ denotes the path between $f(u)$ and $f(v)$ in H with respect to f.

***Definition*2.2.**[3]The *wirelength* of an embedding f of G into H is given by

$$WL_f(G,H) = \sum_{(u,v) \in E(G)} d_H(f(u), f(v)) = \sum_{e \in E(H)} EC_f(G, H(e))$$

where $d_H(f(u), f(v))$ denotes the length of the path $P_f(u, v)$ in H. Then the *wirelength* of G into H is defined as,

$$WL(G,H) = \min WL_f(G,H)$$

where the minimum is taken over all the embeddings.

The *edge isoperimetric problem* is used to solve the wirelength problem when the host graph is a path and is NP-complete.

***Problem* :** Find a subset of vertices of a given graph, such that the number of edges in the subgraph induced by this subset is maximal among all induced subgraphs with the same number of vertices. Mathematically, for a given m, if $I_G(m) = max_{A \subseteq V, |A|=m} |I_G(A)|$ where $I_G(A) = \{(u,v) \in E : u, v \in A\}$, then the problem is to find $A \subseteq V$ and $|A| = m$ such that $I_G(m) = |I_G(A)|$.

***Lemma* 2.3.**(*Congestion lemma*) [3] Let G be an r-regular graph and f be an embedding of G into H. Let S be an edge cut of H such that the removal of edges of S leaves H into 2 components H_1 and H_2 and let $G_1 = f^{-1}(H_1)$ and $G_2 = f^{-1}(H_2)$. Also S satisfies the following conditions:
(i) For every edge $(a, b) \in G_i, i = 1, 2$, $P_f(f(a), f(b))$ has no edges in S.
(ii) For every edge (a, b) in G with $a \in G_1$ and $b \in G_2$, $P_f(f(a), f(b))$ has exactly one edge in S.
(iii) G_1 is a maximum subgraph on k vertices where $k = |V(G_1)|$.
Then $EC_f(S)$ is minimum and $EC_f(S) = rk - 2|E(G_1)|$.

***Lemma* 2.4**.(Partition lemma) [3] Let $f: G \to H$ be an embedding. Let $\{S_1, S_2, ..., S_p\}$ be a partition of $E(H)$ such that each S_i is an edge cut of H. Then,

$$WL_f(G,H) = \sum_{i=1}^{p} EC_f(S_i).$$

3. COMPUTING EXACT WIRELENGTH OF EMBEDDING INCOMPLETE HYPERCUBE INTO GENERALIZED BOOKS

***Definition* 3.1**. [5] For $r \geq 1$, let Q^r denote the graph of r-dimensional hypercube. The vertex set of Q^r is formed by the collection of all r-dimensional binary representations. Two vertices $x, y \in V(Q^r)$ are adjacent if and only if the corresponding binary representations differ exactly in one bit.

Equivalently if $n = 2^r$ then the vertices of Q^r can also be identified with integers 0, 1, ...,$n - 1$ so that if a pair of vertices i and j are adjacent then $i - j = \pm 2p$ for some $p \geq 0$.

***Definition* 3.2**. A set of m vertices of Q^r is said to be a composite set if the number of edges of the subgraph induced by these m vertices is not less than the number of edges of a subgraph induced by any other set of m vertices of Q^r. A composite hypercube of Q^r is defined to be a subgraph of Q^r, which is induced by some composite set of Q^r.

***Definition* 3.3.** An incomplete hypercube on i vertices of Q^r is the subcube induced by {0, 1, …,i-1} and is denoted by L_i, $1 \leq i \leq 2^r$.

Theorem 3.4*. The hybercube can be embedded into* cycle-of-ladders *with minimum wirelength.*

Theorem 3.5*. The hybercube can be embedded into* cycle-of-ladders *with dilation 3.*

4. CONCLUSION

In this paper, we present an algorithm to compute the exact wirelength of embedding the hypercubes into cycle-of-ladders. Embedding of variants of hypercube into cycle-of-ladders are under investigation.

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