NEW ALGORITHM TO FIND THE RELIABILITY OFACONNECTED-(1,2)-OR-(2,1)-OUT-OF-(*m*,*n*):F LINEAR AND CIRCULAR SYSTEM USING MARKOV CHAINS

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Abstract

The connected-(*r*,*s*)-or-(*s*,*r*)-out-of-(*m*,*n*): F linear and circular systems consists of *mn* components, it fails if and only if at least any connected subset of (*r*,*s*)-or-(*s*,*r*) of failed components occurs. For example the connected-(1,2)-or- (2,1)-out-of-(*m*,*n*): F linear and circular systems fails if at least a (1,2)-matrix (i.e. a row with 2 components) or a (2,1)-matrix (a column with two components) of failed components occurs. Many researchers set numerous algorithms to compute the reliability of such systems.

In this paper, a new algorithm to evaluate the reliability of connected-(1,2)-or-(2,1)-out-of-(*m*,*n*): F linear and circular systems. The algorithm depends on representing the functioning states of the system as the states of a suitable Markov chain; this gives the possibility of computing the reliability in terms of the transition probabilities of the considered Markov chain. The new algorithm seems to be much simpler than the existing ones in the literature. Furthermore the computation process of the reliability of the circular system is simpler than the linear system since the number of states of the Markov chain in the circular case is smaller than that of the linear case.

Keywords: Consecutive *k*-out-of-*n*: F system, Connected-*X*-out-of-(*m*,*n*):F system, Markov Chain, Transition probabilities, modular arithmetic (mod), bijection function.

Notations:

I_j^i : The set $\{i, i+1, ..., j\}$

$P(\mathrm{I}_n^1)$: The power set of I_n^1, (The failure space of the components)

$f^{\alpha}(X)$: $f(f.....(f(X)))$ thecomposite function α times.

d_X : The cardinality (number of elements) of the set *X*.

$p_j(q_j)$: The reliability (unreliability) of the j^{th} component, $p_W = \prod_{j \notin W} p_j, q_Z = \prod_{j \in W} q_j : \forall W \subseteq \mathrm{I}_n^1$

$p_j^i(q_j^i)$: The reliability (unreliability) of the j^{th} component at the i^{th} layer (circle).

$R(W)$: The reliability of the consecutive *k*-out-of-*n*: F system when the indices of failed components labeled by the set *W*. $R(W) = p_W q_W$

$R_{L(C)}(m)$: The Reliability of the connected-(1,2)-or-(2,1)-out-of-(*m*,*n*): F linear (circular) system.

1. INTRODUCTION

Kontoleon (1980) [3] was the first studied the consecutive *k*-out-of-*n*: F system that consists of *n* components. Later many researchers have studied the consecutive *k*-out-of-*n*: F systems due to its importance in applications (e.g. telecommunication systems with *n* relay stations, the pipeline of transmit oil system, etc.) where many generalizations are achieved [1], [2], [5], and [6], in these works, the systems are classified according to the connection between components into two types: linear and circular. Such systems fail if at least *k* consecutive components fail.

Boehme *et al*., (1992) [1] have generalized the consecutive *k* out of *n*: F linear and circular system into two dimensional "connected X-out-of-(*m*,*n*): F linear and circular system of (*mn*) components, where the linear system is arranged as a matrix with *m* rowsand *n* columns, while the circular system arranged as *m* circles and *n* rays, (the intersections of circles and rays represent the elements). The two dimensional connected *X*-out-of-(*m*,*n*): F linear and circular system fail, if the connected *X*(*X* may be (*r*,*s*) or *X*=(*r*,*s*)-or-(*s*,*r*) $s, r \leq m, n$) components fail. Also [1] introduced a practical example for such systems, "the supervision system" when (*m*,*n*)=(4,4) as shown in the following Fig.

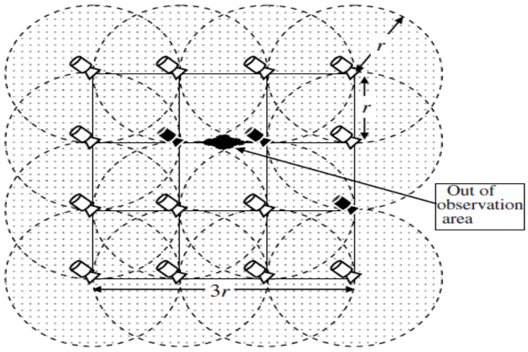

Fig. 1: An out observation area in the (1,2) or (2,1)-out-of (4,4): F linear system

Each TV camera can supervise a disk of radius *r*, and the cameras in each row and column are of the same type and are a distance*r* from each other. The supervision system is failed if an area inside of the sketched square with sides 3*r* is out of observation. The system fails if (at least) two connected cameras (connected by a line) in a row or a column fail. Failed elements are represented by black cameras. The black area between the cameras at the position rows and column respectively (2,2) and (2,3) is out of observation.

Further, investigations regarding the reliability of the connected (*r*,*s*)-out-of-(*m*,*n*): F systems are given by Yamamoto and Myakawa [7] and Yamamoto and Akiba [8]. In 2008, Yamamoto *et al*. [9] achieved a recursive algorithm for evaluating the reliability of a connected-(1,2)-or-(2,1)-out-of-(*m*,*n*):F systems.

In this article, a new algorithm to evaluated the reliability of the connected (1,2) or (2,1)-out-of-(*m,n*): F linear and circular systems is obtained, it depends on representing the function states of the system by a suitable Markov chain.

The following assumptions are assumed to be satisfied by the connected (1,2)-or- (2,1)-out-of-(*m*,*n*): F linear and circular systems.

1. The state of the component and the system is either "functioning" or "failed".
2. All the components are mutually statistically independent.

2. SYMMETRIC PROPERTY IN THE CONSECUTIVE *K*-OUT-OF-*N*: F SYSTEMS

Consider the one-dimensional consecutive *k*-out-of-*n*: F linear and circular system, and $\mathrm{I}_n^1 = \{1,2,...,n\}$ denotes the labels (indices) of the failed components. We shall refer the system using the indices of the failed components, if the system is in the functioning state; we name the set of failed component by the functioning subset, otherwise the failed subset. (For example in the consecutive 2-out-of-*n*: F linear (circular) system, the set $X = \{13\} \subset \mathrm{I}_n^1$ or for simply 13, indicates that the 1st and the 3rd components are failed, and in spite of these failed components but the system still functioning, so we name 13 by a functioning subset. On the contrary of the set $12 \subset \mathrm{I}_n^1$ is a failed subset).

2.1. Symmetric Property of the Consecutive *k*-out-of-*n*: F Linear Systems

Consider the one-dimensional consecutive *k*-out-of-*n*: F linear systems. The event representing the functioning (failure) of the system starting from the 1st component is equivalent to the event of representing the functioning (failure) starting from the last component. This equivalence is valid also, If we replace the first component by the second and the *n*th component by the *n*-1th and so on… .
We shall call this property by the symmetric property between components in the one-dimensional consecutive *k*-out-of-*n*: F linear systems. To represent this property, define $f_L : \mathrm{I}_n^1 \to \mathrm{I}_n^1$ be a bijection function, such that, $f_L(r) = n+1-r$ for any $r \in \mathrm{I}_n^1$. It is easy to see that $f_L = f_L^{-1}$ .Note that $\forall X \subseteq \mathrm{I}_n^1, \alpha \in \mathbf{Z}$

$$f_L^{\alpha}(X) = \begin{cases} X & \alpha \text{ is even} \\ f(X) & \alpha \text{ is odd} \end{cases}$$

Define the following relation between the two subsets $X, Y \in P(\mathrm{I}_n^1)$ such that: $X \sim Y \Leftrightarrow \exists \alpha \in \mathbf{Z} \ni Y = f_L^{\alpha}(X)$

The following lemma shows that the symmetric property defined above for the one-dimensional linear consecutive *k*-out-of-*n*: Fsystems is an equivalence relation. It is well known that the equivalence relation on a set classifies this set into a number of mutually disjoint classes.

Lemma 2.1.1: The relation $(\sim)$ is an equivalence relation.

Proof: *Reflexivity:* $X \sim X \Leftrightarrow X = f_L^{2\alpha}(X) : \forall \alpha$

Transitivity: $X \sim Y \Leftrightarrow \exists \alpha_1 \ni Y = f_L^{\alpha_1}(X), Y \sim Z \Leftrightarrow \exists \alpha_2 \ni Z = f_L^{\alpha_2}(Y) \Rightarrow Z = f_L^{\alpha_2}\left(f_L^{\alpha_1}(X)\right) = f_L^{\alpha_1+\alpha_2}(X)$

Symmetry: $X \sim Y \Leftrightarrow \exists \alpha \in \mathbf{Z} \ni Y = f_L^{\alpha}(X) \Leftrightarrow X = f_L^{\alpha+1}(Y) \Leftrightarrow Y \sim X$.

Define the class of the set *X* using the equivalence relation by $[X]_L^k = \{Y \in P(\mathrm{I}_n^1) : X \sim Y\}$ .If $X = \varnothing$, this implies $f_L^{\alpha}(\varnothing) = \varnothing : \forall \alpha \in \mathbf{Z}$ ($\varnothing$ means that there is no failed components, and we shall denote its class by $[0]_L^k$).

Now, let $\Theta_L^k(\Psi_L^k)$ be the set of all functioning (failed) subsets of I_n^1 for the consecutive *k*-out-of-*n*: F linear system, Θ_L^k may be represented as $\Theta_L^k = \{X \in P(\mathrm{I}_n^1) : \mathrm{I}_{r+k-1}^r \not\subset X;\; r \in \mathrm{I}_{n-k+1}^1\}$.

Lemma 2.1.2: If *X* is a functioning (failed) subset of I_n^1 , and $Y \in [X]_L^k$, then Y is also a functioning (failed) subset of I_n^1 .

Proof: Let *X* is a functioning subset then $\mathrm{I}_{r+k-1}^r \not\subset X \Rightarrow f_L^{\alpha}(\mathrm{I}_{r+k-1}^r) \not\subset f_L^{\alpha}(X) = Y$ such that $Y \in [X]_L^k \Leftrightarrow \exists \alpha \in \mathbf{Z} \ni f_L^{\alpha}(X) = Y$. Now, and if α is even $f_L^{\alpha}(\mathrm{I}_{r+k-1}^r) = \mathrm{I}_{r+k-1}^r \not\subset f_L^{\alpha}(X) = Y$, hence *Y* is functioning subset. If α is odd $f_L^{\alpha}(\mathrm{I}_{r+k-1}^r) = \mathrm{I}_{n+1-r}^{n-(r+k)+2} \not\subset f_L^{\alpha}(X) = Y$, this leads that *Y* is a functioning subset.(The same proof for failed subset, if we put $\subseteq$ instead of $\not\subset$).

In the linear system, and according the above lemma 2.1.2, we conclude that all elements in any class are functioning or failed subset of I_n^1 , hencewe called the class who has any functioning (failed) subset of I_n^1 by a functioning (failed) class.Lemma 2.1.1 shows that $P(\mathrm{I}_n^1) = \Theta_L^k \cup \Psi_L^k$ is a union of a finite partition of mutually disjoint classes consequently Θ_L^k is also may be written as a union of a finite partition of mutually functioning disjoint classes of the form $[X]_L^k = \{Y \in \Theta_L^k : X \sim Y\}$,Ifthe number of these classes is *s*+1, then

$$\Theta_L^k = \{[0]_L^k = [X_0]_L^k, [X_1]_L^k, \ldots, [X_s]_L^k\} = \bigcup_{u=0}^{s} [X_u]_L^k \qquad (2.1.1),$$

Therefore, R_L^k the reliability of the consecutive *k*-out-of-*n*: F linear system can be written as a sum of reliability of the functioning classes. $R_L^k = \sum_{[X_u]_L^k \in \Theta_L^k} R[X_u]_L^k = \sum_{u=0}^{s} R[X_u]_L^k = \sum_{u=0}^{s} \sum_{Z \in [X_u]_L^k} R(Z) = \sum_{u=0}^{s} \sum_{Z \in [X_u]_L^k} p_Z q_Z$,

where $R[X_u]_L^k$ is the reliability of the class $[X_u]_L^k$.

2.2. Symmetric Property of the Consecutive *k*-out-of-*n*: F Circular Systems

The same technique used in section 2.1 will be applied here in the circular case.Consider the one-dimensional consecutive *k*-out-of-*n*: F circular systems. The event representing the functioning (failure) of the system starting from the 1st component is equivalent to the event representing the functioning (failure) starting from any other component in the circle. This equivalence is valid in the clockwise rotation and the contrary direction. We shall call this property by the symmetric property between components in the one-dimensional consecutive *k*-out-of-*n*: F circular systems.

To represent this property, define $f_C : \mathrm{I}_n^1 \to \mathrm{I}_n^1$ be a bijection function, such that $f_C(r) := (r \bmod_n) + 1$ for any $r \in \mathrm{I}_n^1$. Note that $f_C^n(X) = X$ and $f_C^{-\delta} = f_C^{n-\delta}$.

Define a relation between the two subsets $X, Y \in P(\mathrm{I}_n^1)$ such that $X \equiv Y \Leftrightarrow \exists \delta \in \mathbf{Z} \ni f_C^{\delta}(X) = Y$

Lemma 2.2.1*:* The relation $(\equiv)$ is equivalence relation.

Proof: the same proof of lemma 2.1.1.

If $X = \varnothing$ then $f_C^{\alpha}(\varnothing) = \varnothing : \forall \alpha \in \mathbf{Z}$ so we shall denote also its class by $[0]_C^k$. Now define the class of the set *X* using the equivalence relation $[X]_C^k = \{Y \in P(\mathrm{I}_n^1) : X \equiv Y\}$.

Let $\Theta_C^k(\Psi_C^k)$ be the set of all functioning (failed) subsets of I_n^1 for the consecutive *k*-out-of-*n*: F circular system represented using the indices of failed components, Θ_C^k may be represented as:

$$\Theta_C^k = \left\{X : \bigcup_{\alpha=0}^{k-1} f^{\alpha}(r) \not\subset X \subseteq \mathrm{I}_n^1, \forall r \in \mathrm{I}_n^1\right\}$$

Lemma 2.2.2: If X is a functioning (failed) subset, and $Y \in [X]_C^k$, then Y is *also a* functioning (failed) subset.

Proof: if $Y \in [X]_C^k$ then $\exists \alpha \in \mathbf{Z} \ni f_C^\alpha(X) = Y$ and since X is a functioning subset, i.e.; $\bigcup_{\beta=0}^{k-1} f^\beta(r) \not\subset X$, this implies

$$f_C^\alpha\left(\bigcup_{\beta=0}^{k-1} f^\beta(r)\right) = \bigcup_{\beta=0}^{k-1} f^\beta\left(f^\alpha(r)\right) = \bigcup_{\beta=0}^{k-1} f^\beta\left((r \bmod_n)+1\right) \not\subset f_C^\alpha(X) = Y,$$

hence$Y$ is a functioning subset(The same proof for failed subset, if we put $\subseteq$ instead of$\not\subset$).
According the above, lemma 2.2.2, we conclude also that all elements in any class are functioning or failed subset of I_n^1, consequently the class who has any functioning (failed) subset of I_n^1 is called functioning (failed) class.Now, since $P(\mathrm{I}_n^1) = \Theta_C^k \cup \Psi_C^k$, lemma 2.1.1 shows that $P(\mathrm{I}_n^1)$ is a union of a finite partition of mutually disjoint classes as well as Θ_C^k is also may be written as a union of a finite partition of mutually disjoint classes of the form $[X]_C^k = \{Y \in \Theta_C^k : X \equiv Y\}$,If s+1 is the number of these classes, then

$$\Theta_C^k = \left\{[0]_C^k = [X_0]_C^k, [X_1]_C^k, ..., [X_s]_C^k\right\} \bigcup_{u=0}^{s} [X_u]_C^k \qquad (2.2.1)$$

Therefore, R_C^k the reliability of the one-dimensional consecutive k-out-of-n: F circular system can be written as a summation of functioning classes

$$R_C^k = \sum_{[X_u]_C^k \in \Theta_C^k} R[X_u]_C^k = \sum_{u=0}^{s} R[X_u]_C^k = \sum_{u=0}^{s} \sum_{Z \in [X_u]_C^k} R(Z) = \sum_{u=0}^{s} \sum_{Z \in [X_u]_C^k} p_Z q_Z,$$

Where $R[X_u]_C^k$ is the reliability of the class $[X_u]_C^k$.

Lemma 2.2.3:if the components in the consecutive k-out-of-n: F linear (circular) system are *i.i.d.*,and $Z \in [X]_{L(C)}^k$,then $R(Z) = R(X)$.

Proof: If $Z \in [X]_{L(C)}^k \Leftrightarrow \exists \alpha \ni Z = f_{L(C)}^\alpha(X)$, since $f_{L(C)}^\alpha$ is a bijection function, i.e.; $d_Z = d_X$ and since the components are *i.i.d.* , i.e; $p_j = p : j = 1,2,...,n$, then $R(Z) = p_Z q_Z = p^{n-d_Z} q^{d_Z} = p^{n-d_X} q^{d_X} = p_X q_X = R(X)$.

In the following section, we shall specialize the relations and partitions given the case of the one-dimensional consecutive 2-out-of-n: F linear and circular systems, to compute the reliability of the two-dimensional connected (1,2) or (2,1)-out-of-(m,n): F linear and circular systems.

3. A MARKOV CHAIN TECHNIQUE FOR CALCULATING THE RELIABILITY OF (1,2) OR (2,1)-OUT-OF-(m,n): F LINEAR AND CIRCULAR SYSTEMS.

Consider the connected-(1,2)-or-(2,1)-out-of-(m,n): F linear (circle) system, and let $\mathrm{I}_n^1 = \{1,2,...,n\}$ be the indices of the components in the i^{th} layer (circle), then the failed components of any layer (circle) may be represented as an element of $P(\mathrm{I}_n^1)$. If system is in the functioning state, then any layer (circle) has a functioning subsets of the consecutive 2-out-of-n: F linear (circular) system $\Theta_{L(C)}^2$, otherwise we have at least two connected failed components, which implies that the whole system fails. Let $\Theta_{L(C)}^2\left(\Psi_{L(C)}^2\right)$ be the set of all functioning (failure) subsets of I_n^1 for any layer (circle) in the considered systemand $[X_{s+1}]_{L(C)}^2 = \Psi_{L(C)}^2$, thenequation 2.1.1 in the linear system (equation 2.2.1 in the circular system) are implying that

$$P(\mathrm{I}_n^1) = \Theta_{L(C)}^2 \cup \Psi_{L(C)}^2 = \left(\bigcup_{j=0}^{s} [X_j]_{L(C)}^2\right) \cup [X_{s+1}]_{L(C)}^2 = \bigcup_{j=0}^{s+1} [X_j]_{L(C)}^2 \qquad (3.1)$$

Where $[X_j]_{L(C)}^2 : j = 0,1,...,s$ is a partition of mutual disjoint the functioning classes of $\Theta_{L(C)}^2$, and $[X_{s+1}]_{L(C)}^2$ is the only failed class.

Consider $X \in \Theta_{L(C)}^2$ represents a functioning subset in the i^{th} layer (circle), define $\Theta_{L(C)}^2(X)$ to be the set of allfunctioning subset labeled by the set $Z \in \Theta_{L(C)}^2$ in the(i+1) th layer (circle) that guarantee that there is no common failed components with X i.e.; Z must not intersect with any failed components of the functioning subset X in the i^{th} layer (circle), otherwise the whole system fails, i.e.; $\Theta^2_{L(C)}(X)=\left\{Z\in\Theta^2_{L(C)}: Z\cap X=\varnothing\right\}$. (Note that $\Theta^2_{L(C)}(0)=\Theta^2_{L(C)}$). Also, define $A^{i+1}_{L(C)}\left(X,[Y]^2\right)=\left\{Z\in[Y]^2_{L(C)}: Z\cap X=\varnothing\right\}$ be the set of all elements from the class $[Y]^2_{L(C)}$ in the $(i+1)^{th}$ layer (circle) that guarantee that the failed components of Z must be not connected (intersect) to the failed components of the subset X, i.e. the intersection between the $\Theta^2_{L(C)}(X)$ and the class $[Y]^2_{L(C)}$ (note that $A^{i+1}_{L(C)}\left(0,[Y]^2\right)=[Y]^2_{L(C)}, A^{i+1}_{L(C)}\left(Y,[0]^2\right)=\{0\}$).

Lemma 3.1: $f^{\alpha}_{L(C)}(X_u)=W:\alpha\in\mathbf{Z}\Rightarrow f^{\alpha}_{L(C)}\left(A^{i+1}_{L(C)}\left(X_u,[Y]^2\right)\right)=A^{i+1}_{L(C)}\left(W,[Y]^2\right)$.

Proof: If $Z\in f^{\alpha}_{L(C)}\left(A^{i+1}_{L(C)}\left(X_u,[Y]^2\right)\right)\Leftrightarrow\exists H\in A^{i+1}_{L(C)}\left(X_u,[Y]^2\right)\subseteq[Y]^2\ni Z=f^{\alpha}_{L(C)}(H)$

In the circular system $f^{n-\alpha}_C(Z)=H\ni H\cap X_u=\varnothing\Leftrightarrow$

$f^{\alpha}_C(H\cap X_u)=f^{\alpha}_C(H)\cap f^{\alpha}_C(X_u)\underset{f^{\alpha}(X_u)=W}{=}Z\cap W=\varnothing\Leftrightarrow Z\in A^{i+1}_C\left(W,[Y]^2\right)$

In the linear system $f^{\alpha}_L(Z)=H\ni H\cap X_u=\varnothing\Leftrightarrow f^{\alpha}_L(H\cap X_u)=f^{\alpha}_L(H)\cap f^{\alpha}_L(X_u)$

$f^{\alpha}_L(H)\cap f^{\alpha}_L(X_u)\underset{f^{\alpha}(X_u)=W}{=}Z\cap W=\varnothing\Leftrightarrow Z\in A^{i+1}_L\left(W,[Y]^2\right)\Leftrightarrow Z\in A^{i+1}_{L(C)}\left(W,[Y]^2\right)$.

Definition 3.1: The connected (1,2) or (2,1) out of (m,n): F linear (circular) system could be Imbedded in a Markov chain (see Koutras [4]) as follows:

a) The state of any layer (circle) is represented by an element of the failure space of components $P\left(\mathrm{I}^1_n\right)$, and without loss of generality, we can rearrange $P\left(\mathrm{I}^1_n\right)=\bigcup_{j=0}^{s+1}[X_j]^2_{L(C)}$ where $[X_u]^2_{L(C)}\cap[X_v]^2_{L(C)}=\varnothing:\forall u\neq v$.

b) If the subsystem with i layers (circles) in the functioning state, and we need to add a new layer (circle) to the subsystem and keep it in the functioning state. So the failed components in the $(i+1)^{th}$ layer (circle) depends only on the failed components on the i^{th} layer (circle) and must be not connected to those in the i^{th} layer (circle). If $S_i\in P\left(\mathrm{I}^1_n\right): i=1,2,...,m$ is a random variable represents the state of the subsystem with i layer (circle), then the random variable S_{i+1} depends only on S_i but not on $S_{i-1},S_{i-2},...,S_1$, hence the sequence $\{S_i\}, i=1,2,...,m$ forms a Markov chain such that:

I. The variables $S_i: i=1,2,...,m$ are defined on $P\left(\mathrm{I}^1_n\right)$ such that, $S_i\in[X_u]^2_{L(C)}$ u=0,1,…,s. If and only if the i^{th} layer (circle) in the system with i layers (circles) has the failed components labeled of any set from the class $[X_u]^2_{L(C)}$ i.e.; has reached the $[X_u]^2_{L(C)}$ level of deterioration.

II. $S_i\in[X_{s+1}]^2_{L(C)}$ if the subsystem consisting of i layers (circles) is failed.

Theorem 3.1: Consider the connected (1,2) or (2,1)-out-of-(i,n): F linear (circular) subsystem, $S_i\in P\left(\mathrm{I}^1_n\right): i=1,2,...,m$, the variable S_i representing the state of the i^{th} layer (circle) as in definition 3.1, then

1. If $P_{L(C)}[X_u,X_v]$ is the transition probabilities that the system moves from the state $[X_u]^2_{L(C)}$ with i layers (circles) to the state $[X_v]^2_{L(C)}$ with i+1 layers (circles) and is given by:

$$P_{L(C)}[X_u,X_v]=\begin{cases}\dfrac{\sum\limits_{W\in[X_u]^2_{L(C)}}\left(p^i_W q^i_W\sum\limits_{Z\in A^{i+1}_{L(C)}\left(Z,[X_v]^2\right)}p^{i+1}_Z q^{i+1}_Z\right)}{\sum\limits_{W\in[X_u]^2_{L(C)}}p^i_W q^i_W} & u,v=0,1,2,...s\\ 1-\sum\limits_{v=0}^{s}P_{L(C)}[X_u,X_v] & u=0,1,2,...s, v=s+1\\ 0 & u=s+1, v=0,1,...,s\\ 1 & u=s+1, v=s+1\end{cases}$$

2. The reliability of system can be expressed as: $R_{L(C)}(m) = \sum_{[X_v]^2_{L(C)} \in \Theta^2_{L(C)}} P^m_{L(C)}[0, X_v] = \sum_{v=0}^{s} P^m_{L(C)}[0, X_v]$

where $P^m_{L(C)}[0, X_u]$ is the *m*-step transition probability.

3. If the components are *i.i.d.*, and $C_{X_u} = 1 - \sum_{v=0}^{s} d_u^{X_v} p^{n-d_{X_v}} q^{d_{X_v}}, d_{X_u}^{X_v} = d_{A^{i+1}_{L(C)}\left(X_u, [X_v]^2\right)}$,

(note that $d^0_{X_u} = d_{A^{i+1}_{L(C)}\left(X_u, [0]^2\right)} = d_0 = 1; u = 0,1,...,s$), Then the probability transient matrix is expressed as:

$$\mathbf{P}_{L(C)} = \begin{array}{c|ccccccc|c}
\text{classes} & [0]^2_{L(C)} & [X_1]^2_{L(C)} & [X_2]^2_{L(C)} & & [X_l]^2_{L(C)} & & [X_s]^2_{L(C)} & \Psi^2_{L(C)} \\
\hline
[0]^2_{L(C)} & d_0^0 p^n & d_0^{X_1} p^{n-d_{X_1}} q^{d_{X_1}} & d_0^{X_2} p^{n-d_{X_2}} q^{d_{X_2}} & \cdots & d_0^{X_l} p^{n-d_{X_l}} q^{d_{X_l}} & \cdots & d_0^{X_s} p^{n-d_{X_s}} q^{d_{X_s}} & C_0 \\
[X_1]^2_{L(C)} & d_{X_1}^0 p^n & d_{X_1}^{X_1} p^{n-d_{X_1}} q^{d_{X_1}} & d_{X_1}^{X_2} p^{n-d_{X_2}} q^{d_{X_2}} & \cdots & d_{X_1}^{X_l} p^{n-d_{X_l}} q^{d_{X_l}} & \cdots & d_{X_1}^{X_s} p^{n-d_{X_s}} q^{d_{X_s}} & C_{X_1} \\
[X_2]^2_{L(C)} & d_{X_2}^0 p^n & d_{X_2}^{X_1} p^{n-d_{X_1}} q^{d_{X_1}} & d_{X_2}^{X_2} p^{n-d_{X_2}} q^{d_{X_2}} & \cdots & d_{X_2}^{X_l} p^{n-d_{X_l}} q^{d_{X_l}} & \cdots & d_{X_2}^{X_s} p^{n-d_{X_s}} q^{d_{X_s}} & C_{X_2} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
[X_l]^2_{L(C)} & d_{X_l}^0 p^n & d_{X_l}^{X_1} p^{n-d_{X_1}} q^{d_{X_1}} & d_{X_l}^{X_2} p^{n-d_{X_2}} q^{d_{X_2}} & \cdots & d_{X_l}^{X_l} p^{n-d_{X_l}} q^{d_{X_l}} & \cdots & d_{X_l}^{X_s} p^{n-d_{X_s}} q^{d_{X_s}} & C_{X_l} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
[X_s]^2_{L(C)} & d_{X_s}^0 p^n & d_{X_s}^{X_1} p^{n-d_{X_1}} q^{d_{X_1}} & d_{X_s}^{X_2} p^{n-d_{X_2}} q^{d_{X_2}} & & d_{X_s}^{X_l} p^{n-d_{X_l}} q^{d_{X_l}} & \cdots & d_{X_s}^{X_s} p^{n-d_{X_s}} q^{d_{X_s}} & C_{X_s} \\
\hline
\Psi^2_{L(C)} & 0 & 0 & 0 & & 0 & \cdots & 0 & 1
\end{array}_{(s+2)\times(s+2)}$$

Proof:

1. If the subsystem with *i* layer (circle) is represented by the state $[X_u]^2_{L(C)}$ andthe (*i*+1)th layer (circle) is represented by the state $[X_v]^2_{L(C)}$ *u, v* =0,1,… , *s*, where*i*=1,2,…,*m*-1, then

$$P_{L(C)}[X_u, X_v] = \frac{P\left(S_{i+1} = [X_v]^2_{L(C)}, S_i = [X_u]^2_{L(C)}\right)}{P\left(S_i = [X_u]^2_{L(C)}\right)} = \frac{\sum_{W \in [X_u]^2_{L(C)}} \left(p_W^i q_W^i P\left\{ A^{i+1}_{L(C)}\left(W, [X_v]^2\right) \right\} \right)}{\sum_{W \in [X_u]^2_{L(C)}} p_W^i q_W^i}$$

$$= \frac{\sum_{W \in [X_u]^2_{L(C)}} \left(p_W^i q_W^i \sum_{Z \in A^{i+1}_{L(C)}\left(W, [X_v]^2\right)} p_Z^{i+1} q_Z^{i+1} \right)}{\sum_{W \in [X_u]^2_{L(C)}} p_W^i q_W^i}$$

For $u = 0,1,2,...s$, and $v = s+1$, $P_{L(C)}[X_u, X_v]$ indicates that the subsystem moves from the functioning state $[X_u]^2_{L(C)}$ to the failure state $[X_v]^2_{L(C)}$, using Markov properties $P_{L(C)}[X_u, X_{s+1}] = 1 - \sum_{v=0}^{s} P_{L(C)}[X_u, X_v]$. For $u = s+1$, $P_{L(C)}[X_u, X_v]$ indicates that the system breakdown, this level corresponds to an absorbing state, hence $P_{L(C)}[X_{s+1}, X_v]$ equal 1 for $v = s+1$ and 0 otherwise.

2. For any *i*=1,2,..,*m*-1, if $[X_u]^2_{L(C)} = [0]^2_{L(C)}$ then;

$$\sum_{v=0}^{s} P_{L(C)}[0, X_v] = \sum_{v=0}^{s} \frac{P\left(S_{i+1} = [X_v]^2_{L(C)}, S_i = [0]^2_{L(C)}\right)}{P\left(S_i = [0]^2_{L(C)}\right)} = \sum_{v=0}^{s} \frac{\left(p^i_{\varnothing}\right) P\left\{A_{L(C)}\left(0, [X_v]^2_{L(C)}\right)\right\}}{\left(p^i_{\varnothing}\right)}$$

$$= \sum_{v=0}^{s} P\left\{A_{L(C)}\left(0, [X_v]^2_{L(C)}\right)\right\} = \sum_{v=0}^{s} P\left(S_{i+1} = [X_v]^2_{L(C)}\right) = P\left\{\Theta^2_{L(C)}\right\} = R^2_{L(C)} = R_{L(C)}(1)$$

According theorem 3.1 in [4], if $\boldsymbol{\pi}_0^T = [1 \quad 0 \quad ..... \quad 0]$ is the initial probability, and $\mathbf{u} = [1 \quad 1 \quad \cdots \quad 1 \quad 0]^T$ is the row vector, then the reliability of the connected (1,2) or (2,1)-out-of-(*m*,*n*): F linear (circular) system is

$$R_{L(C)}(m)=\boldsymbol{\pi}_0^T\left(\prod_{i=1}^{m}\mathbf{P}\right)\mathbf{u}=\sum_{v=0}^{s}P_{L(C)}^{m}[0,X_v]=\sum_{[X_v]_{L(C)}^2\in\Theta_{L(C)}^2}P_{L(C)}^{m}[0,X_v]=R_{L(C)}(m)$$

If the components are *i.i.d.* $p_j^i=p:\forall j=1,2,\dots,n;\forall i=1,2,\dots,m$ and

$$P_{L(C)}[X_u,X_v]=\frac{\sum_{W\in[X_u]_{L(C)}^2}\left(p_W^i q_W^i\sum_{Z\in A_{L(C)}^{i+1}(W,[X_v]^2)}p_Z^{i+1}q_Z^{i+1}\right)}{\sum_{W\in[X_u]_{L(C)}^2}p_W^i q_W^i}$$

Using lemma 2.2.3, $R(W)=R(X_u)=p^{n-d_{X_u}}q^{d_{X_u}},R(Z)=R(X_v)=p^{n-d_{X_v}}q^{d_{X_v}}$

$$P_{L(C)}[X_u,X_v]=\frac{\left(\sum_{W\in[X_u]_{L(C)}^2}p^{n-d_{X_u}}q^{d_{X_u}}\right)\left(\sum_{Z\in A_{L(C)}^{i+1}(W,[X_v]^2)}p^{n-d_{X_v}}q^{d_{X_v}}\right)}{\sum_{W\in[X_u]_{L(C)}^2}p^{n-d_{X_u}}q^{d_{X_u}}}=\frac{\sum_{W\in[X_u]_{L(C)}^2}\left(p^{n-d_{X_u}}q^{d_{X_u}}\sum_{Z\in A_{L(C)}^{i+1}(W,[X_v]^2)}p^{n-d_{X_v}}q^{d_{X_v}}\right)}{\sum_{W\in[X_u]_{L(C)}^2}p^{n-d_{X_u}}q^{d_{X_u}}}=\sum_{Z\in A_{L(C)}^{i+1}(W,[X_v]^2)}p^{n-d_{X_v}}q^{d_{X_v}}$$

Since $\forall W\in[X_u]_{L(C)}^2\Rightarrow\exists\alpha\in\mathbf{Z}\ni f_{L(C)}^{\alpha}(X_u)=W$, then using lemma 3.1.

$$P_{L(C)}[X_u,X_v]=\sum_{Z\in A_{L(C)}^{i+1}(W,[X_v]^2)}p^{n-d_{X_v}}q^{d_{X_v}}=\sum_{f^{\alpha}(Z)\in f_{L(C)}^{\alpha}\left(A_{L(C)}^{i+1}(W,[X_v]^2)\right)=A_{L(C)}^{i+1}(X_u,[X_v]^2)}p^{n-d_{X_v}}q^{d_{X_v}}=d_{X_u}^{X_v}p^{n-d_{X_v}}q^{d_{X_v}}$$

4. ALGORITHM

In this section, we propose an algorithm for calculating the reliability of a connected-(1,2)-or-(2,1)-out-of-(*m*,*n*): F linear and circular systems. This algorithm is based on the applications of the results given by *Lemma* 2.1.1, *Lemma* 2.2.1 and *Lemma* 3.1. We illustrate the steps of the proposed algorithm using the following example:

Example 1: Computing the reliability of the connected (1,2) or (2,1)-out-of-(2,4): F Linear System.

1. (the partition $\Theta_{L(C)}^2$ and failed space $\Psi_{L(C)}^2$).Setting the functioning states of consecutive 2-out-of-4: F linear system $\Theta_L^2=\{0,1,2,3,4,13,14,24\}$

The state of M.C. is $[0]_L^2=\{0\},[1]_L^2=\{1,4\},[2]_L^2=\{2,3\},[13]_L^2=\{13,24\},[14]_L^2=\{14\},\Psi_L^2$

2. Calculate the transient probability matrix $P_{L(C)},P_{L(C)}^m$.

$$\mathbf{P}_L=\begin{pmatrix}\text{classes}&[0]_L^2&[1]_L^2&[2]_L^2&[13]_L^2&[14]_L^2&\Psi_L^2\\ [0]_L^2&p^4&2p^3q&2p^3q&2p^2q^2&p^2q^2&C_0\\ [1]_L^2&p^4&p^3q&2p^3q&p^2q^2&0&C_1\\ [2]_L^2&p^4&2p^3q&p^3q&p^2q^2&p^2q^2&C_2\\ [13]_L^2&p^4&p^3q&p^3q&p^2q^2&0&C_{13}\\ [14]_L^2&p^4&0&2p^3q&0&0&C_{14}\\ \Psi_L^2&0&0&0&0&0&1\end{pmatrix}$$

for example $A_L^{i+1}(2,[13])=\left\{Z\in[13]_L^2:\{2\}\cap Z=0\right\}=\{13\}\Rightarrow P_L[2,13]=p^2q^2$

3. The reliability of the connected-(1,2)-or-(2,1)-out-of-(*m*,*n*): F linear (circular) system is

$$R_{L(C)}(m)=\sum_{[X_v]_{L(C)}^2\in\Theta_{L(C)}^2}P_{L(C)}^m[0,X_v]$$

$$R_L(2)=\sum_{[Y]_L^2\in\Theta_L^2}P_L^2[0,Y]=\left[p^8+4p^7q+3p^6q^2\right]+\left[2p^7q+6p^6q^2+2p^5q^3\right]+\left[2p^7q+6p^6q^2+4p^5q^3\right]+\left[2p^6q^2+4p^5q^3+2p^4q^4\right]$$
$$+\left[p^6q^2+p^5q^3\right]=p^8+8p^7q+18p^6q^2+12p^5q^3+2p^4q^4$$

Example 2: Computing the reliability of the connected (1,2) or (2,1)-out-of-(3,6): F circular system

<u>Step 1:</u> Setting the functioning space of consecutive 2-out-of-6: F circular system

$$\Theta_C^2 = \{0,1,2,3,4,5,6,13,14,15,24,25,26,35,36,46,135,246\}$$

$[X_0]_C^2 = \{0\}$, $[X_1]_C^2 = [1]_C^2 = \{1,2,3,4,5,6\}$, $[X_2]_C^2 = [13]_C^2 = \{13,24,35,46,15,26\}$, $[X_3]_C^2 = [14]_C^2 = \{14,25,36\}$, $[X_4]_C^2 = [135]_C^2 = \{135,246\}$

The states of the Markov Chain $[0]_C^2, [1]_C^2, [13]_C^2, [14]_C^2, [135]_C^2, \Psi_C^2$

<u>Step 2:</u> Calculating P_C, P_C^3 for example

$$\mathbf{P}_C = \begin{pmatrix} \text{classes} & [0]_C^2 & [1]_C^2 & [13]_C^2 & [14]_C^2 & [135]_C^2 & \Psi_C^2 \\ [0]_C^2 & p^6 & 6p^5q & 6p^4q^2 & 3p^4q^2 & 2p^3q^3 & C_0 \\ [1]_C^2 & p^6 & 5p^5q & 4p^4q^2 & 2p^4q^2 & p^3q^3 & C_1 \\ [13]_C^2 & p^6 & 4p^5q & 3p^4q^2 & p^4q^2 & p^3q^3 & C_{13} \\ [14]_C^2 & p^6 & 4p^5q & 2p^4q^2 & 2p^4q^2 & 0 & C_{14} \\ [135]_C^2 & p^6 & 3p^5q & 3p^4q^2 & 0 & p^3q^3 & C_{135} \\ \Psi_C^2 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$A_C^{i+1}(1,[14]) = \left\{ Z \in [14]_C^2 : \{1\} \cap Z = 0 \right\} = \{25,36\} \Rightarrow d_1^{14} = 2 \Rightarrow P_C[1,14] = 2p^4q^2$$

Step 3: Calculating the reliability of the system

$$R_C(3) = \sum_{[Y]_C^2 \in \Theta_C^2} P_C^3[0,Y]$$

$$\begin{aligned} R_C(3) = {} & p^{18} + 12p^{17}q + 48p^{16}q^2 + 76p^{15}q^3 + 48p^{14}q^4 + 12p^{13}q^5 + 2p^{12}q^6 + \\ & 6p^{17}q + 66p^{16}q^2 + 240p^{15}q^3 + 342p^{14}q^4 + 192p^{13}q^5 + 42p^{12}q^6 + 6p^{11}q^7 + \\ & 6p^{16}q^2 + 66p^{15}q^3 + 240p^{14}q^4 + 342p^{13}q^5 + 192p^{12}q^6 + 42p^{11}q^7 + 6p^{10}q^8 \\ & 2p^{15}q^3 + 18p^{14}q^4 + 54p^{13}q^5 + 66p^{12}q^6 + 36p^{11}q^7 + 12p^{10}q^8 + 2p^9q^9 \end{aligned}$$

$$R_C(3) = p^{18} + 18p^{17}q + 123p^{16}q^2 + 408p^{15}q^3 + 705p^{14}q^4 + 642p^{13}q^5 + 308p^{12}q^6 + 84p^{11}q^7 + 18p^{10}q^8 + 2p^9q^9$$

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