SPANNING TREES IN CIRCULANT NETWORKS

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Abstract

A spanning tree of a connected graph G is defined as a maximal set of edges of G that contains no cycle, or as a minimal set of edges that connect all vertices. The Maximum Leaf Spanning Tree (MLST) problem is to find a spanning tree of a graph with as many leaves as possible. In general case, the problem is proved to be NP-hard. In this paper we find a spanning tree with maximum number of leaves for the circulant graph $\mathrm{G}(\mathrm{n}, \pm\{1,2\})$.

1. INTRODUCTION

A leaf in a graph *G* of a spanning tree is a vertex in *G* of degree one in the spanning tree. Given a graph *G*, the Maximum Leaf Spanning Tree problem is to find the maximum number of vertices of degree one in a spanning tree over all spanning trees of *G*. Some broadcasting problems in network design ask to minimize the number of broadcasting nodes, which must be connected to a single root. This translates the problem into finding a spanning tree with many leaf and few internal nodes. Ongoing research on this topic is motivated by the fact that variants of this problem occur frequently in real life applications [1,2,3]. The Maximum Leaf Spanning Tree (MLST) problem can be found in the area of communication networks and circuit layouts [1]. The communication networks where the vertices correspond to terminals and a problem on message routing is to design a tree-like layout in the network, “leaf terminals” may have lighter workloads than “intermediate terminals” of degree at least two. Hence, in this case, the solution of MLSTP problem provides a reasonable layout [3].

The Maximum Leaf Spanning Tree problem is equivalent to the Minimum Connected Dominating Set problem, where one should find a minimum set of vertices $S \subseteq V$ of the input graph *G* such that the subgraph of *G* induced by *S* is connected and *S* is a dominating set of *G*. Connected Dominating Set is a fundamental problem in connected facility location problem and is studied intensively in computer science and operations research [4, 5]. It is also a central problem in wireless networking [2,6,8]. A connected dominating set (CDS) service as a virtual backbone of a network which can help with routing. Any vertex outside the virtual backbone can send message or signal to another vertex through the virtual backbone. We may impose a virtual backbone to support shortest path routing, fault-tolerant routing, multi-casting, radio broadcasting, etc [7,8]. Furthermore, a virtual backbone of a wireless network may reduce communication overhead, increase bandwidth efficiency, and decrease energy consumption [9].

The Maximum Leaf Spanning Tree (MLST) problem and Minimum Connected Dominating Set problem are known to be NP-hard [10]; many literature references discussed the MCDS & MLST problem by approximation algorithms [11,12,13,14,15,16,17]. The problems are discussed for some of trapezoid graphs and generalized trapezoid in [17], grid graph in [13], Hypercube and Star network in [14].

Even though there are numerous results and discussions on MLST and MCDS problems, most of them deal with only approximate results. In this paper we produce a Minimum Connected Dominating Set for the circulant network $G(n, \pm\{1,2\})$, thereby solving the Maximum Leaf Spanning Tree problem.

2. CIRCULANT NETWORK

The circulant network is a natural generalization of double loop network, which was first considered by Wong and Coppersmith [21]. Circulant graphs have been used for decades in the design of computer and telecommunication networks due to their optimal fault-tolerance and routing capabilities [20]. It is also used in VLSI design and distributed computation [26,27]. Theoretical properties of circulant graphs have been studied extensively and surveyed by Bermond et al. [22]. Every circulant graph is a vertex transitive graph and a Cayley graph [24]. Most of the earlier research concentrated on using the circulant graphs to build interconnection networks for distributed and parallel systems [20,22]. Classes of graphs that are circulant graphs include the complete graphs, complete bipartite graphs, Paley graphs of prime order, prism graphs, möbius ladder graph, tetrahedral graph and torus grid graphs.

Definition 1[25]: A circulant undirected graph denoted by $G\left(n;\pm\{1,2,3....,j\}\right)$, $1 < j \leq n/2, n \geq 3$ is defined as an undirected graph consisting of the vertex set *V* = {0,1, ….., *n*-1} and the edge set *E* = { (*i*, *j*): $|j - i| \equiv s(\text{mod } n)$, $s \in \{1, 2, ….., j\}$.

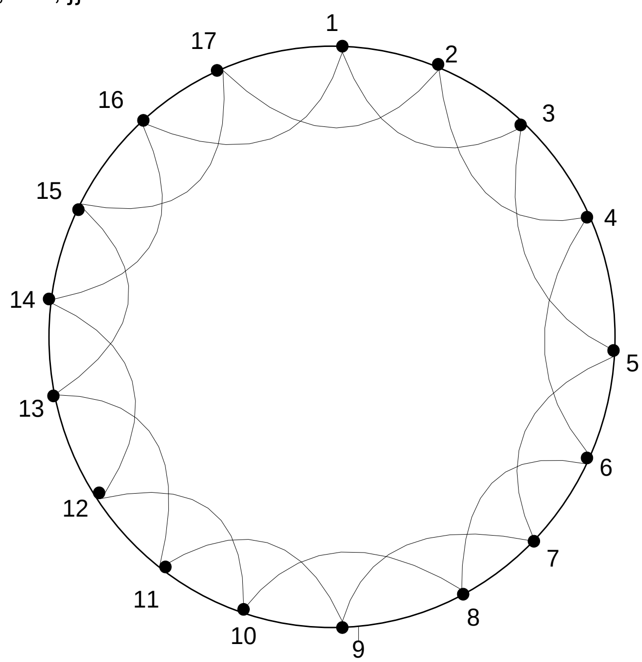

Fig. 1: Circulant Network $G(17, \pm\{1,2\})$

It is clear that $G(n, \pm 1)$is the undirected cycle C_n and $G(n, \pm\{1,2, \dots, n/2\})$ is the complete graph K_n.

In this paper we investigate the particular case of the Circulant Network namely $G(n, \pm\{1,2\})$

3. SPANNING TREE FOR $G(n, \pm\{1,2\})$

Let $G(V, E)$be a simple, undirected graph. Given a vertex $v \in V$, the set of its neighbors is defined by $N(v) = \{u \in V \mid (u, v) \in E\}$. A subtree *T* of *G* is a spanning tree of *G* if V_T = V.

The following two problems on the spanning trees of a graph $G(V, E)$ have been considered in the literature [17] and are NP-complete [10].

Problem 1: Given an undirected graph *G* = (*V*,*E*), $|V| = n, |E| = m$, find a spanning tree *T* of *G* with a maximum number of leaves $T_l(G)$.

Problem 2: Given a connected graph $G(V,E)$ and a vertex set $R \subseteq V$ R is a Dominating Set if each vertex in G is either in R or has at least one neighbour in R. A Dominating Set with minimum cardinality is called Minimum Dominating Set. The vertex domination problem is to determine a minimum Dominating Set with an independent set of vertices. We denote the cardinality of a minimum Dominating Set by $\gamma(G)$. R is a Connected Dominating Set if R is a Dominating Set and the induced subgraph $G[R]$ is connected. A connected dominating set with minimum cardinality is called a minimum connected dominating set and its cardinality be defined by $\gamma_c(G)$.

Remark: Maximum Leaf Spanning Tree problem is equivalent to the Minimum Connected Dominating Set problem [17].

The following results are easy to observe.

Lemma 1: Let G be an r - regular graph on n vertices,

$$\text{then } \gamma(G) \geq \left\lceil \frac{n}{r+1} \right\rceil.$$

Lemma 2: Let G be the circulant graph $G(n, \{\pm 1, 2\}$,

$$\text{then } \gamma(G) \geq \left\lceil \frac{n}{5} \right\rceil.$$

Theorem 1: The Minimum Connected Domination number of $G(n, \pm\{1,2\})$

$$\gamma_c(G) \geq 1 + \left\lceil \frac{n-5}{2} \right\rceil.$$

3.1 Domination Algorithm

Input: circulant graph $G(n, \pm\{1,2\})$

Algorithm: Label the vertices of graph $G(n, \pm\{1,2\})$ as 1,2,…,n in the clockwise sense. select the vertices labeled 1,5,9,13…,n-3 if $n \equiv 0 \; mod \; 4$; else select the vertices labeled 1,5,9,13…,$4\left\lfloor \frac{n}{4} \right\rfloor - 3, n-4$, $n \not\equiv 0 \; mod \; 4$. Include all the edges incident at each of these vertices.

Output: Spanning tree with maximum number of leaves for $G(n, \pm\{1,2\})$

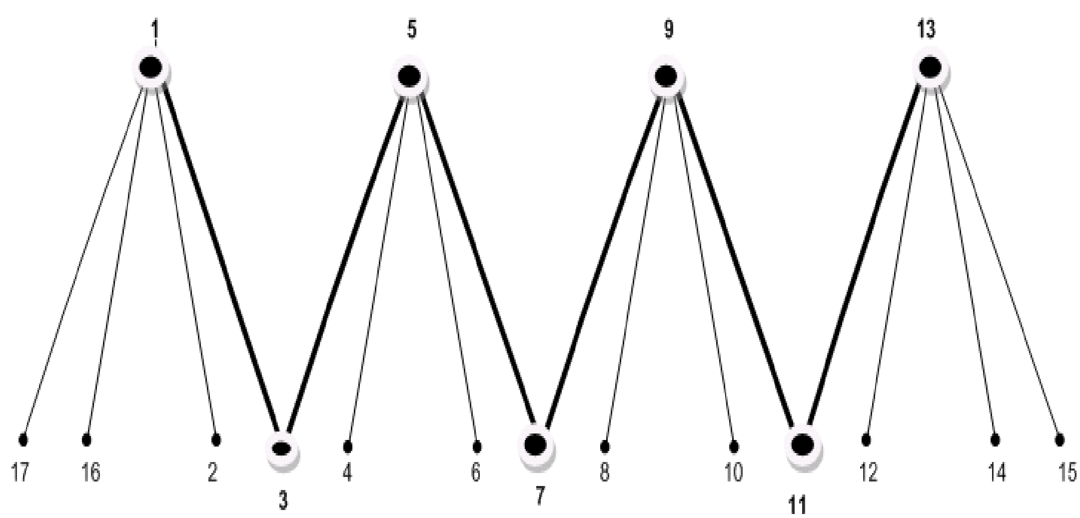

Fig. 2: MLST for $G(17, \pm\{1,2\})$

Proof of correctness: Vertices labeled 1 dominate 4 vertices. The vertex labeled 5 dominate one of these vertices 4 vertices and dominate 3 other vertices. This is true for vertex labeled 9,13,….. when 1,5,9,13…,*n*-3. When $n \equiv 0 \; mod \; 4$ the vertex labeled *n*-3 dominates the rest of the vertices, but fewer than 3 vertices. When $n \not\equiv 0 \; mod \; 4$ the vertex labeled *n*-4 dominates the rest of the vertices, but fewer than 3 vertices. In either case, the set of vertices with degree of each vertex greater then 1 induce a path and constitute a connected dominating set. But the cardinality of this set is $1 + \left\lceil \frac{n-5}{2} \right\rceil$.

The proofs of the following theorems are easy consequence of MLST Algorithm

Theorem 2: The The Minimum Connected Domination number of $G(n, \pm\{1,2\})$

$$\gamma_c(G) = 1 + \left\lceil \frac{n-5}{2} \right\rceil$$

Theorem 3: The Maximum Number of leaves for $G(n, \pm\{1,2\})$

$$\mathrm{T}_l(G) = \left\lceil \frac{n}{2} \right\rceil + 1.$$

3. CONCLUSION & FUTURE WORK

In this paper we have been discussed the Maximum Leaf Spanning Tree problem and Minimum Connected Dominating Set problem of the circulant network $G(n, \pm\{1,2\})$. The Minimum Connected Dominating Set problem for the circulant network $G(n, \pm\{1,2,\dots,j\})$ is under consideration.

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