CUTWIDTH OF CIRCULANT NETWORKS

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Abstract

The cutwidth problem of a graph G is to embed G into a path such that the maximum number of edges crossing along any cut in the path is minimized. In this paper we prove the cutwidth of Circulant networks.

Keywords - circulant graphs, cutwidth, minimum cut, layout, Minimum Cut Arrangement problem, graph layout.

1 INTRODUCTION

Given a graph with *n* vertices, a layout on the graph is a bijection between the vertex set and the set of natural numbers from 1 to *n*. The cutwidth of a layout is the maximum number of edges connecting vertices on opposite sides of any of the “gaps” between successive vertices in the linear layout. Computing cutwidth is an NP-complete problem[8], even when the graphs are restricted to planar graphs of maximum degree three[1], unit disk graphs[13], partial grids, and consequently bipartite graphs. The problem of computing the cutwidth of a graph is known as the *Minimum Cut Arrangement* problem[4]. This important graph layout problem was first proposed as a model to minimize the number of channels in a circuit[2,9], and more recently it has found applications in areas like protein engineering[6], network reliability[5], automatic graph drawing[20], information retrieval[21], and as a subroutine in the cutting plane algorithm for TSP[14].

Polynomial-time algorithms for the exact computation of cutwidth are known only for few graph classes. Exact results are proved for various standard graphs such as paths, cycles, stars, complete graphs, hypercubes, complete bipartite graphs and graph products, 2,3-dimensional meshes, 2, 3-dimensional toroidal meshes, cylindrical 2-dimensional meshes[16], complete *k*-level *t*-ary trees, trees with diameter at most 4 and double stars, mobius ladders,caterpillars, and (*m*, *n*)-multipath[17]. Cutwidth of proper interval graphs has a trivial solution following an interval ordering of the vertices[18].

The circulant graphs are an important class of topological structures of interconnection networks. They are symmetric with simple structures and easy extendability. Circulant graphs have been used for decades in the design of computer and telecommunication networks due to their optimal fault-tolerance and routing capabilities[10]. The circulant network is a natural generalization of a double loop network, which was first considered by Wong and Coppersmith[7]. Circulant graphs also constitute the basis for designing certain data alignment networks for complex memory systems. Most of the earlier research concentrated on using the circulant graphs to build interconnection networks for distributed and parallel systems. Circulant graphs are intensively researched in computer science, graph theory and discrete mathematics.

2 PRELIMINARIES

In this section we give the basic definitions and preliminaries related to cutwidth problems.

Given an undirected graph $G = (V, E)$ with $|V| = n$, a layout φ on G is a one-to-one function $\varphi : V \rightarrow \{1, 2, 3, \ldots, n\}$. Given a layout φ on G, we define the sets $L(i, \varphi, G) = \{u \in V(G) : \varphi(u) \leq i\}$, $R(i, \varphi, G) = \{u \in V(G): \varphi(u) > i\}$ and the cut $\Theta(i, \varphi, G) = |\{uv \in E(G) : u \in L(i, \varphi, G) \cap v \in R(i, \varphi, G)\}|$.

Definition 2.1. [12] Given a graph $G = (V, E)$, and a layout φ on G, let $cw(\varphi,G)=\max_{i=1 to\ n} \Theta(i, \varphi, G)$. Define $cw(G)= min_{\varphi} cw(\varphi, G)$, where the minimum is taken over all linear layouts φ on G. The cutwidth problem of G is to find a layout φ on G such that $cw(\varphi, G)= cw(G)$. The cutwidth problem is NP-complete[9].

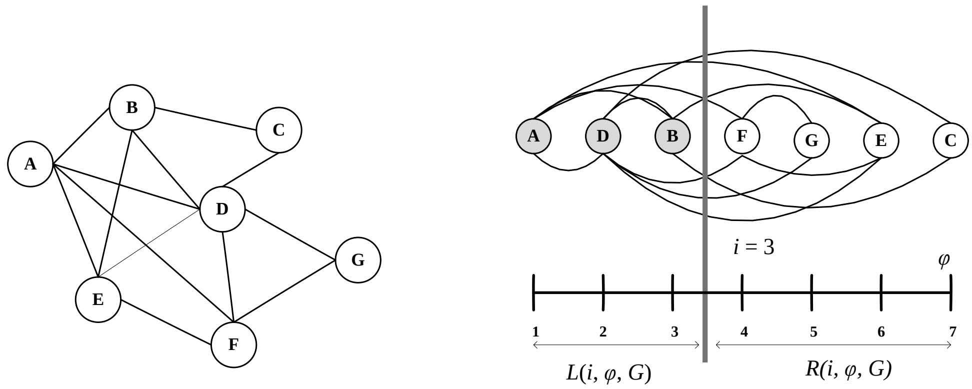

Figure 1: Graphical representation of the layout $\varphi = \{(A, 1), (B, 3), (C, 7), (D, 2), (E, 6), (F, 4),(G, 5)\}$.

In Figure 1, by drawing a vertical line between position 3 and position 4, the vertices to the left of the line belong to $L(3, \varphi, G)$ and the vertices to the right of the line belong to $R(3, \varphi, G)$. It is easy to compute the cut $\Theta(3, \varphi, G)$ by counting the number of edges that cross the vertical line, in other words the number of edges that cross from $L(3, \varphi, G)$ to $R(3, \varphi, G)$.

Definition 2.3. [15] A circulant undirected graph, denoted by $G(n; \pm S)$ where $S \subseteq \{1, 2, \ldots, \lfloor n/2 \rfloor\}$, $n \geq 3$ is defined as a graph consisting of the vertex set $V =\{0, 1, \ldots, n-1\}$ and the edge set $E = \{(i, j) : |j - i| \equiv s (\text{mod } n), s \in S\}$.

For convenience we label $V(G)$ as 1, 2, 3, ..., n. The circulant graph shown in Figure 2 (a) is $G(8; \pm\{1, 2, 3\})$. It is clear that $G(n; \pm 1)$is the undirected cycle C_n and $G(n; \pm\{1, 2, \ldots, \lfloor n/2 \rfloor\})$ is the complete graph K_n.

Further $G(n; \pm\{1, 2, 3, \ldots, j\})$, $1 \leq j < \lfloor n/2 \rfloor$, $n \geq 3$ is a $2j$ -regular graph[11].

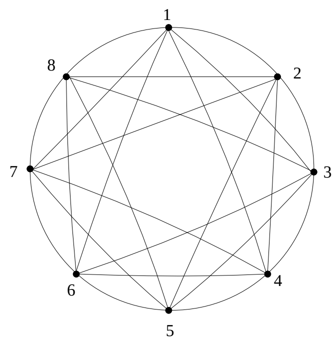

Figure 2: $G(8; \pm \{1, 2, 3\})$

3 EXACT CUTWIDTH OF CIRCULANT GRAPH

In this session we prove the cutwidth of Circulant networks.

Problem 3.1. [19] (Maximum subgraph problem) A maximum subgraph problem for a graph $G = (V, E)$ is to find an induced subgraph A of G on k vertices for a given integer k, with maximum number of edges of G.

Lemma 3.2. [19] (Congestion lemma) Let G be an r-regular graph and f be an embedding of G into H. Let S be an edge cut of H such that the removal of edges of S leaves H into 2 components H_1 and H_2 and let $G_1 = f^{-1}(H_1)$ and $G_2 = f^{-1}(H_2)$. Also S satisfies the following conditions:

(i) For every edge $(a, b) \in G_i$, $i = 1, 2$, $P_f(f(a), f(b))$ has no edge in S.

(ii) For every edge (a, b) in G with $a \in G_1$ and $b \in G_2$, $P_f(f(a), f(b))$ has exactly one edge in S.

(iii) G_1 is a maximum subgraph on k vertices where $k = |V(G_1)|$.

Then $EC_f(S)$ is minimum and $EC_f(S) = rk - 2|E(G_1)|$.

Lemma 3.3. [11] A set of k consecutive vertices of $G(n; \pm1)$induces a maximum subgraph of $G(n; \pm S)$ on k vertices, $k \leq n/2$, $S=\{1, 2, 3, ..., j\}$,$1 \leq j < n/2$, $n \geq 3$.

Lemma 3.4. [11] The number of edges in a maximum subgraph on k vertices of $G(n; \pm S)$, $S = \{1, 2, 3, . . . , j\}$, $1 \leq j < \lfloor n/2 \rfloor$, $1 \leq k \leq n$, $n \geq 3$ is given by

$$\xi = \begin{cases} \frac{k(k-1)}{2} & k \leq j+1 \\ kj - \frac{j(j+1)}{2} & j+1 < k \leq n-j \\ \left\{(n-k)^2 + (4j+1)k - (2j+1)n\right\} & n-j < k \leq n \end{cases}$$

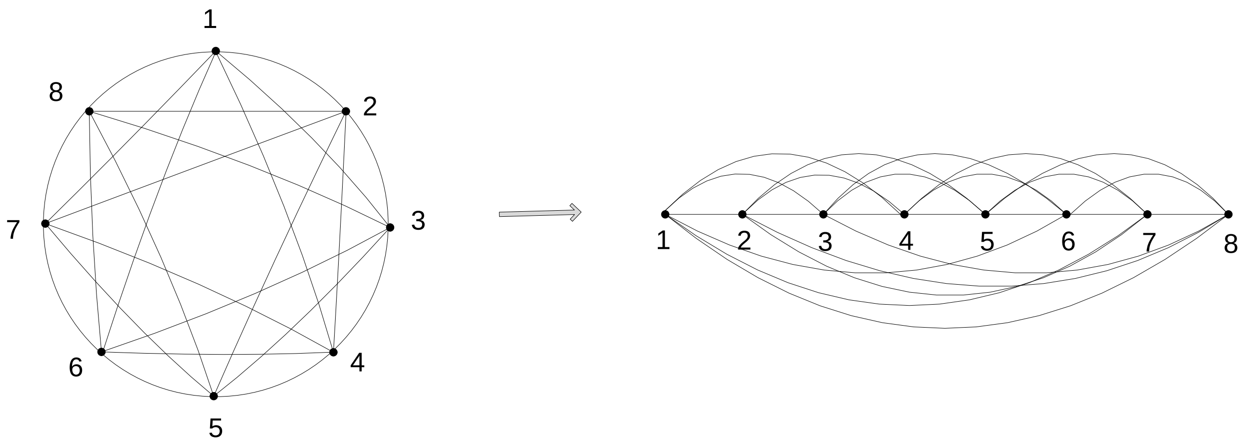

Figure 3: Embedding of $G(8;\pm \{1, 2, 3\})$ in $P_{8.}$

Theorem 3.5. Let $G(n; \pm S)$, $S = \{1, 2, 3, . . . , j\}$, $1 \leq j < \lfloor n/2 \rfloor$, $1 \leq k \leq n$, $n \geq 3$ be a circulant graph with n vertices. Then the cutwidth of the given graph G is given by $cw(G) = j(j+1)$.

Proof. Let $G(n; \pm S)$ be a circulant graph with n vertices and $S = \{1, 2, 3, . . . , j\}$, $1 \leq j < \lfloor n/2 \rfloor$, $1 \leq k \leq n$, $n \geq 3$. Let H be the maximal subgraph with k vertices $(k < n)$ in G. By Lemma 3.3, The set of k consecutive vertices of G induces a maximum subgraph on k vertices, $k \leq n/2$. By Lemma 3.4, the number of edges in a maximum subgraph on *k* vertices of *G*(*n*; ±*S*), *S* = {1, 2, 3, . . . , *j*}, 1 ≤*j* < $\lfloor n/2 \rfloor$, 1 ≤ *k* ≤ *n*, *n* ≥ 3 is given by

$$\xi = \begin{cases} \frac{k(k-1)}{2} & k \leq j+1 \\ kj - \frac{j(j+1)}{2} & j+1 < k \leq n-j \\ \left\{(n-k)^2 + (4j+1)k - (2j+1)n\right\} & n-j < k \leq n \end{cases}$$

Case (i) $k \leq j+1$

Hence number of edges $\xi = \frac{k(k-1)}{2}$. By Conjestion lemma[19]Cutwidth at a point $k \leq j+1$ is given by *cw* = *j*(*j*+1).

Case (ii) $j+1 < k \leq n-j$

Hence number of edges $\xi = kj - \frac{j(j+1)}{2}$. By Conjestion lemma[19]Cutwidth at a point $k > j+1$ is given by *cw* = *j*(*j*+1).

Case (iii) $n-j < k \leq n$

Since it is a regular graph, the cutwidth at the path is symmetrical. Therefore *cw* = *j*(*j*+1).

4 CONCLUSION

In this paper, we compute the exact cutwidth of Circulant graphs. It would be interesting to study the cutwidth of interconnection networks such as torus, cube connected cycle, generalized fat trees and so on.

5 REFERENCES

[1] B. Monien and I. H. Sudborough, Min cut is NP-complete for edge weighted trees, Theoretical Computer Science, vol. 58(1–3), pp. 209–229, 1988.
[2] D. Adolphson and T. C. Hu, Optimal linear ordering, SIAM J. Appl. Math., vol. 25(3), pp. 403–423, 1973.
[3] D. M. Thilikos, M. J. Serna, and H. L. Bodlaender, Cutwidth I: A linear time fixed parameter algorithm, Journal of Algorithms, vol. 56, pp. 1–24, 2005.
[4] D. M. Thilikos, M. J. Serna, and H. L. Bodlaender, Cutwidth II: Algorithms for partial w-trees of bounded degree, Journal of Algorithms, vol. 56, pp. 24–49, 2005
[5] D. R. Karger, A randomized fully polynomial approximation scheme for the all-terminal network reliability problem, SIAM Journal on Computing, vol. 29, pp. 492–514, 1999.
[6] G. Blin, G. Fertin, D. Hermelin and S. Vialette, Fixed-parameter algorithms for protein similarity search under mRNA structure constraints, Journal of Discrete Algorithms, vol. 6, pp. 618–626, 2008.
[7] G. K. Wong and D. A. Coppersmith, A combinatorial problem related to multimodule memory organization, Journal of Association for Computing Machinery, vol. 21, pp. 392-401, 1974.
[8] F. Gavril, Some NP-complete problems on graphs, Proceedings of 11th Conference on Information Sciences and Systems, Baltimore, pp. 91–95, 1977.
[9] F. Makedon and I. H. Sudborough, On minimizing width in linear layouts, Discrete Applied Mathematics, vol. 23, pp. 243–265, 1989.
[10] F. T. Boesch and J. Wang, Reliable Circulant Networks with Minimum Transmission Delay, IEEE Transactions on Circuit and Systems, Vol. 32, pp. 1286-1291, 1985.
[11] I. Rajasingh, P. manuel, M. Arockiaraj and B. Rajan, Embeddings Of Circulant Networks, Journal of Combinatorial Optimization, 2011.

[12] J. Diaz, J. Petit and M. Serna, A Survey Of Graph Layout Problems, ACM Computing Surveys, vol. 34, pp. 313–356, 2002.
[13] J. Diaz, M. D. Penrose, J. Petit, and M. J. Serna, Layout problems on lattice graphs in Computing and Combinatorics, Lecture Notes in Computer Science, vol. 1627, pp. 103–112, 1999.
[14] J. Petit, Addenda to the Survey of Layout Problems, Bulletin of the EATCS, No.105, 2011.
[15] J. M. Xu, Topological Structure and Analysis of Interconnection Networks, Kluwer Academic Publishers, Netherlands, 2001.
[16] J. Rolim, O. Sykora, and I. Vrto, Optimal cutwidths of meshes, Lecture Notes in Computer Science, vol. 1017, pp. 252–264, 1995.
[17] L. Yixun, L. Xianglu and Y. Aifeng, A degree sequence method for the cutwidth problem of graphs, Appl. Math. J. Chinese Univ. Ser B, vol. 17(2), pp. 125-134, 2002.
[18] P. Heggernes, D. Lokshtanov, R. Mihai and C. Papadopoulos, Cutwidth of split graphs, threshold graphs, and proper interval graphs, Vol 5344, Lecture Notes in Computer Science, pp.218–229, 2008.
[19] P. Manuel, I. Rajasingh, B. Rajan and H. Mercy, Exact wirelength of hypercube on a grid, Discrete Appl. Math. vol. 157(7), pp. 1486-1495, 2009.
[20] P. Mutzel, A polyhedral approach to planar augmentation and related problems. In Proceedings of ESA 1995, LNCS 979, pp. 497–507, Springer, 1995.
[21] R. A. Botafogo, Cluster analysis for hypertext systems. In Proceedings of SIGIR, ACM, pp. 116–125, 1993.