SOLUTION OF LANE-EMDEN EQUATION BY RESEDUAL POWER SERIES METHOD

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Abstract

This paper investigates the approximate solution of the Lane-Emden equation using new analytic technique. The solution was calculated in the form of a convergent power series with easily computable components. The proposed method obtains Taylor expansion of the solution and reproduces the exact solution when the solution is polynomial. The proposed technique is applied to several examples to illustrate the accuracy, efficiency, and applicability of the method.

Keywords - Lane-Emden equation, residual power series method, Taylor expansion.

1 INTRODUCTION

The Lane-Emden equation has been used to model several phenomena in mathematical physics and astrophysics such as the theory of stellar structure, the thermal behavior of a spherical cloud of gas, the isothermal gas spheres, and the theory of thermionic currents [1-8]. The Lane-Emden equation was first studied by the astrophysicists Jonathan Homer Lane and Robert Emden, where they considered the thermal behavior of a spherical cloud of gas acting under the mutual attraction of its molecules and subject to the classical laws of thermodynamics. The reader is kindly requested to go through [1-12] in order to know more details about the Lane-Emden equation, including its history, types and kinds, method of solutions, its applications, and so forth.

In the present paper, we introduce a new simple analytical method we call it the residual power series (RPS) method [13] to find out series solutions to strongly linear and nonlinear Lane-Emden equation. The RPS method is effective and easy to use for solving linear and nonlinear Lane-Emden equation without linearization, perturbation, or discretization. This method constructs an analytical approximate solution in the form of a polynomial. The RPS method is different from the traditional higher order Taylor series method. The Taylor series method is computationally expensive for large orders and suited for the linear problems. The RPS method is an alternative procedure for obtaining analytic Taylor series solution of the Lane-Emden equation. By using residual error concept, we get a series solution, in practice a truncated series solution.

The RPS method has the following characteristics; first, the method obtains Taylor expansion of the solution; as a result, the exact solution is available when the solution is polynomial. Moreover the solutions and all its derivatives are applicable for each arbitrary point in the given interval. Second, the method can be applied directly to the given problem by choosing an appropriate value for the initial guess approximation without any modification. Third, the RPS method needs small computational requirements with high precision and less time.

The purpose of this paper is to obtain symbolic approximate RPS solutions for the Lane-Emden equation of the following form:

$$y''(x) + \frac{\kappa}{x} y'(x) + f\big(y(x)\big) = 0, x \in (0, a), \tag{1}$$

subject to the initial conditions

$$y(0) = \alpha_0, y'(0) = \alpha_1, \tag{2}$$

where f is nonlinear analytic function, $y(x)$ is an unknown function of independent variable x to be determined, and $\alpha_i, \kappa, a \in \mathbb{R}$ with $a > 0$.

In general, the Lane-Emden equation does not always have solutions which we can obtain using analytical methods. Due to this, some authors have proposed numerical methods to approximate the solution. To mention a few, the Adomian decomposition method has been applied to solve Eqs. (1) and (2) as described in [9]. In [10] the author has developed the linearization method to solve the singular Eqs. (1) and (2). In [11] also, the author has provided the variational iteration method to further investigation to Eqs. (1) and (2). Furthermore, the homotopy perturbation method is carried out in [12] for solving singular Eqs. (1) and (2).

However, previous studies require more effort to achieve the results and usually they are suited for linear form of Eqs. (1) and (2). But on the other aspects as well, the applications of other versions of series solutions to linear and nonlinear problems can be found in [13-18] and for numerical solvability of different categories of singular differential equations one can consult the reference [19].

The outline of the paper is as follows: in the next section, we present the basic idea of the RPS method. In section 3, numerical examples are given to illustrate the capability of proposed method. This article ends in section 4 with some concluding remarks.

2 FORMULATION OF THE RESEDUAL POWER SERIES METHOD

In this section, we employ our technique of the RPS method to find out series solution for the Lane-Emden equation subject to the given initial conditions.

The RPS method consists in expressing the solution of Eqs. (1) and (2) as a power series expansion about the initial point $x = 0$ [13]. To achieve our goal, we suppose that this solution takes the form $y(x) = \sum_{m=0}^{\infty} y_m(x)$, where y_m are terms of approximations and are given as $y_m(x) = c_m x^m$.

Obviously, when $m = 0{,}1$ since $y_0(x), y_1'(x)$ satisfy the initial conditions (2) as $y(0) = c_0 = y_0(0)$ and $y'(0) = c_1 = y_1'(0)$, we have the initial guess approximation of $y(x)$ which is as follows:

$$y_{\text{initial}}(x) = y(0) + y'(0)x.$$

If we choose $y_{\text{initial}}(x)$ as initial guess approximation of $y(x)$, then we can calculate $y_m(x)$ for $m = 2,3,\ldots$ and approximate the solution $y(x)$ of Eqs. (1) and (2) by the kth-truncated series

$$y^k(x) = \sum_{m=0}^{k} c_m x^m. \tag{3}$$

Prior to applying the RPS method, we rewrite singular Eqs. (1) and (2) in the form of the following:

$$x\, y''(x) + \kappa y'(x) + x f\big(y(x)\big) = 0. \tag{4}$$

The substituting of kth-truncated series $y^k(x)$ of Eq. (3) into Eq. (4) leads to the following definition for the kth residual function:

$$\mathrm{Res}^k(x) = \sum_{m=2}^{k} m(m-1)c_m x^{m-1} + \kappa \sum_{m=1}^{k} m c_m x^{m-1} + x f\left(\sum_{m=0}^{k} c_m x^m\right), \tag{5}$$

and the following ∞th residual function:

$$\mathrm{Res}^\infty(x) = \lim_{k\to\infty} \mathrm{Res}^k(x).$$

It easy to see that, $\mathrm{Res}^\infty(x) = 0$ for each $x \in [0, a]$. This show that $\mathrm{Res}^\infty(x)$ is infinitely many times differentiable at $x = 0$. On the other hand, $\frac{d^{k-1}}{dx^{k-1}}\mathrm{Res}^\infty(0) = \frac{d^{k-1}}{dx^{k-1}}\mathrm{Res}^k(0) = 0$. In fact, this relation is a fundamental rule in RPS method and its applications [13].

Now, in order to obtain the 2nd-approximate solution, we put $k = 2$ and $y^2(x) = \sum_{m=0}^{2} c_m x$. On the other hand, we differentiate both sides of Eq. (5) with respect to x and substitute $x = 0$, to conclude

$$\frac{d}{dx}\mathrm{Res}^2(0) = 2c_2(1+\kappa) + f(c_0). \tag{6}$$

Using the fact that $\frac{d}{dx}\mathrm{Res}^\infty(0) = \frac{d}{dx}\mathrm{Res}^2(0) = 0$ make Eq. (6) gives the following value for c_2:

$$c_2 = -\frac{1}{2(1+\kappa)} f(c_0).$$

Thus, using 2nd-truncated series, the 2nd-approximate solution for Eqs. (1) and (2) can be written as

$$y^2(x) = \alpha_0 + \alpha_1 x - \frac{1}{2!}\frac{1}{1+\kappa} f(\alpha_0) x^2.$$

Similarly, we can find the 3ed-approximate solution, by letting $k = 3$ and $y^3(x) = \sum_{m=0}^{3} c_m x^m$. On the other aspects as well, this procedure can be repeated till the arbitrary order coefficients of RPS solutions for Eqs. (1) and (2) are obtained. Moreover, higher accuracy can be achieved by evaluating more components of the solution.

Theorem 2.1. Suppose that $y(x)$ is the exact solution for Eqs. (1) and (2). Then, the approximate solution obtained by the RPS method is just the Taylor expansion of $y(x)$.

Proof. See [13].

Corollary 2.1. If $y(x)$ is a polynomial, then the RPS method will be obtained the exact solution.

Proof. Obvious.

It will be convenient to have a notation for the error in the approximation $y(x) \approx y^k(x)$. Accordingly, we will let $\mathrm{Rem}^k(x)$ denote the difference between $y(x)$ and its kth Taylor polynomial obtained from the RPS method; that is,

$$\mathrm{Rem}^k(x) = y(x) - y^k(x) = \sum_{m=k+1}^{\infty} \frac{y^{(m)}(0)}{m!} x^m .$$

The functions $\mathrm{Rem}^k(x)$ is called the kth remainder for the RPS approximation of $y(x)$. In fact, it often happens that the remainders $\mathrm{Rem}^k(x)$ become smaller and smaller, approaching zero, as k gets large.

3 NUMERICAL EXPERIMENTS

The proposed method provides an analytical approximate solution in terms of an infinite power series. However, there is a practical need to evaluate this solution, and to obtain numerical values from the infinite power series. The consequent series truncation and the practical procedure are conducted to accomplish this task, transforms the otherwise analytical results into an exact solution, which is evaluated to a finite degree of accuracy.

In this section, we consider three examples to demonstrate the performance and efficiency of the present technique. Throughout this paper, all the symbolic and numerical computations performed by using Mable 13 software package.

Example 3.1. Consider the following linear nonhomogeneous Lane-Emden equation:

$$y''(x) + \frac{2}{x} y'(x) + y(x) = x^3 + x^2 + 12x + 6, 0 < x < \infty, \quad (7)$$

subject to the initial conditions

$$y(0) = 0, y'(0) = 0. \quad (8)$$

As in the previous discussion, if we select the two terms of approximations as $y_0(x)$ =0 and $y_1(x) = 0$, then the power series expansion of the solution takes the form

$$y(x) = c_2 x^2 + c_3 x^3 + c_4 x^4 + \cdots. \quad (9)$$

Consequently, the 3ed-order approximation of the RPS solution for Eqs. (7) and (8) according to these terms is as follows: $y^3(x) = x^3 + x^2$. It easy to discover that the each of the coefficients c_m for $m \geq 4$ in the expansions (9) are vanished. In other words, $\sum_{m=0}^{\infty} c_m x^m = \sum_{m=0}^{3} c_m x^m$. Thus, the analytic approximate solution of Eqs. (7) and (8) agree well with the exact solutions $y(x) = x^3 + x^2$ with full agreement with Corolary 2.1.

Example 3.2. Consider the following nonlinear homogeneous Lane-Emden equation:

$$y''(x) + \frac{8}{x}y'(x) + 9\pi y(x) + 2\pi y(x)\ln y(x) = 0, 0 < x < \infty, \tag{10}$$

subject to the initial conditions

$$y(0) = 1, y'(x) = 0. \tag{11}$$

Assuming that the initial guess approximation has the form $y^1(x) = 1$. Then, the 10th-truncated series of the RPS solution of $y(x)$ for Eqs. (10) and (11) is as follows:

$$y^{10}(x) = 1 - \frac{\pi}{2}x^2 + \frac{\pi^2}{8}x^4 - \frac{\pi^3}{48}x^6 + \frac{\pi^4}{384}x^8 - \frac{\pi^5}{3840}x^{10} = \sum_{j=0}^{5}\left(-\frac{\pi}{2}\right)^j \frac{x^{2j}}{j}.$$

Thus, the exact solution of Eqs. (10) and (11) has the general form which is coinciding with the exact solution $y(x) = \sum_{j=0}^{\infty}\left(-\frac{\pi}{2}\right)^j \frac{(x^2)^j}{j} = e^{-\frac{\pi}{2}x^2}$.

Let us now carry out the error analysis of the RPS method for this example. Figure 1 shows the exact solution $y(x)$ and the four iterates approximations $y^k(x)$ for $k = 5,10,15,20$. This graph exhibits the convergence of the approximate solutions to the exact solutions with respect to the order of the solutions.

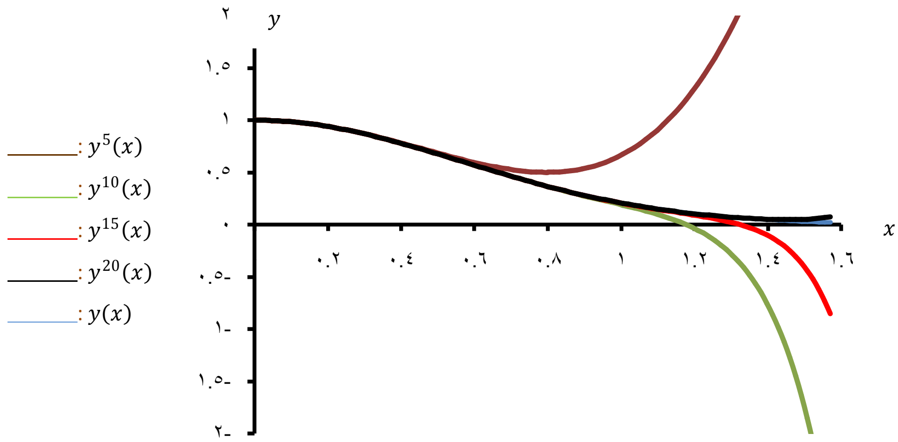

Figure 1. Plots of RPS solution for Eqs. (10) and (11) when $k = 5,10,15,20$ together with exact solution on $\left[0, \frac{\pi}{2}\right]$.

In Figure 2, we plot the error functions $\mathrm{Ext}^k(x)$ when $k = 5,10,15,20$ which are approaching the axis $y = 0$ as the number of iterations increase. This graph shows that the exact errors are getting smaller as the order of the solutions is increasing. In other words, as we progress through more iterations. These error indicators confirm the convergence of the RPS method with respect to the order of approximations.

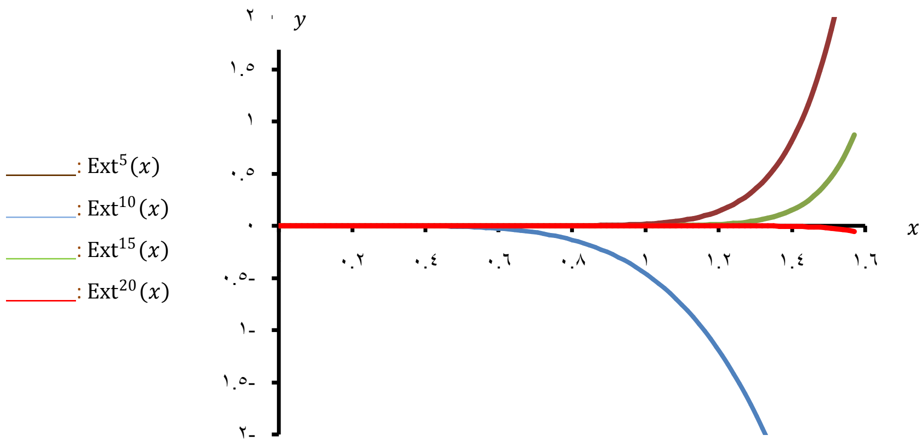

Figure 2. Plots of exact error functions for Eqs. (10) and (11) when $k = 5,10,15,20$ on $\left[0, \frac{\pi}{2}\right]$.

Example 3.3. Consider the following nonlinear homogeneous Lane-Emden equation:

$$y''(x) + \frac{2}{x} y'(x) + 4\left(2e^{y(x)} + e^{\frac{1}{2}y(x)}\right) = 0, 0 < x < \infty, \quad (12)$$

subject to the initial conditions

$$y(0) = 0, y'(0) = 0. \quad (13)$$

As we mentioned earlier, if we select the two terms of approximations as $y_0(x) = 0$ and $y_1(x) = 0$, then the first few terms approximations of the RPS solution for Eqs. (12) and (13) are $y_2(x) = -2, y_3(x) = 0, y_4(t) = 1, y_5(x) = 0, y_5(x) = -\frac{2}{3}$, ..., and so on. If we collect these results, then the 10th-truncated series of the RPS solution for $y(x)$ is as follows:

$$y^{10}(x) = -2x^2 + x^4 - \frac{2}{3}x^6 + \frac{1}{2}x^8 - \frac{2}{5}x^{10} = -2\left((x^2)^1 - \frac{1}{2}(x^2)^2 + \frac{1}{3}(x^2)^3 - \frac{1}{4}(x^2)^4 + \frac{1}{5}(x^2)^5\right).$$

Thus, the exact solution of Eqs. (12) and (13) has the general form which is coinciding with the exact solution $y(x) = -2\sum_{j=1}^{\infty}(-1)^{j+1}\frac{x^{2j}}{j} = -2\ln(1 + x^2)$.

4 CONCLUSION

The main concern of this work has been to propose an efficient algorithm for the solutions of the Lane-Emden equation. The main goal has been achieved by introducing the RPS method to solve this class of singular differential equations. We can conclude that the RPS method is powerful and efficient technique in finding approximate solution for linear and nonlinear Lane-Emden equation.

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