INDEX NUMBER MATHEMATICS APPLICATION TO THE GROSS DOMESTIC PRODUCT DECOMPOSITION

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Abstract

In the focus of this paper is a new methodological approach to upgrading the statement of Gross domestic product (GDP) growth rates and implicit GDP deflators – on annual and quarterly bases. For a long time in the practice of National statistical agencies the chain-linking methodology has been used. By means of chain linking, index number drift has been resolved partially in the sense of the second best solution. As time passes Laypeyres index with fixed base substantially overestimates Paasche index as further as index base is being left in the past. Paasche price index is lower compared to its Laspeyres counterpart but it is the most appropriate GDP deflator due to statistical (Cauchy theorem) and economic (substitution-transformation effect) reasons. Relying on index numbers' theoretical considerations of so called "superlative indices" the authors unanimously chose Törnqvist and Fisher index from all superlative indices as superior one. Superlative indices satisfy the most theoretical and practical requirements relevant for GDP compilation. Lloyd-Moulton index has been also calculated because the key point was econometric estimation of elasticity of substitution. The complete estimation procedure has been carried out on the case study of Croatia. Data base dealing with Croatian Quarterly GDP data has been come with the scope from q1 2000 to q4 2007. Lloyd-Moulton index is not superlative but it measures substitution in the best way which can be estimated by various econometric applications. Besides it possesses the most important theoretical property of superlative indices – exact decomposition. Putting together Lloyd-Moulton with Törnqvist and Fisher indices authors have constructed Lloyd-Moulton-Törnqvist-Fisher (LMTF) model. LMTF model improves GDP price-volume decomposition due to more precise substitution measurement. Fisher index supported by LMTF model has been also built and it resolves the problem of additive (absolute and relative) inconsistency in GDP data. Another significant achievement of the paper is keeping product test identity (volume = volume times price). An integral part of the survey are testing results which prove that Fisher index supported by LMTF model can be considered as "ideal" in the practical applications.

Keywords - Törnqvist, Fisher and Lloyd-Moulton (LMTF) model, Fisher index supported by LMTF model, Gross domestic product (GDP) decomposition, superlative indices, elasticity of substitution, additive GDP consistency

1 SHORT OVERVIEW OF THE INDEX NUMBER THEORY IN THE CONTEXT OF GROSS DOMESTIC PRODUCT DECOMPOSITION

In this paper a comprehensive literature overview was used to provide persuasive arguments for selecting Lloyd Moulton (LM), Törnqvist (T) and Fisher (F) indices as the most appropriate components of Lloyd-Moulton-Törnqvist-Fisher (LMTF) index.

1.1 Index number theory – economic and mathematical approach

Usage of Lloyd-Moulton-Törnqvist-Fisher (LMTF) index in national accounts practice, as and attempt to improve GDP volume-price decomposition by means of classic chain-linking, is supported by numerous mathematical settings. During ex-post methodological revisions, symmetric and particularly "superlative" indices, as a special subcategory of symmetric indices, have been preferred. Here the authors offer those parts of index number literature which prove powerful arguments that Lloyd Moulton (LM), Törnqvist (T) and Fisher (F) indices in constructing LMTF model are superior against other competing indices (e.g. Walsh index). Namely, three fundamental "fruits" of index number theory are: a) theoretical symmetric indices (Laspeyres) L and (Paasche) P indices of Fisher-Shell type - on the production side of GDP - {Fisher-Shell, [6], 1972} and of Konüs type - on the expenditure side of GDP - {Konüs,[11], 1924} are bounded from below or from up by their counter parting empirical L and P indices; or they bound the latter.

b) Diewert {Diewert, [5], 2002, pp. 324} proved that all "superlative indices" (T and F are of this type) converge to each other up to the second order in the sufficiently small neighbourhood (so called Diewert's quadratic approximation lemma). Their first and second direct and cross partial derivatives converge to each other as well, irrespective of which order they are, and

c) theoretical LM index, although it is not from "superlative indices class", is exact decomposable as any "superlative" index.

Diewert {see Diewert, [3], 1983} showed that there is coefficient α ($0 \leq \alpha \leq 1$) which defines feasible sets for the optimisation of output (*R*) and intermediary consumption (*C*), in the sense shown in equations (1) and (2):

$$R\,(\boldsymbol{p},\alpha) = max_q\{\sum_{n=1}^{N} p_n^t q_n : \boldsymbol{q} \in (1-\alpha)\,S^0(u^0) + \alpha\,S^1(u^1)\} \tag{1}$$

$$C\,(\boldsymbol{p},\alpha) = min_q\{\sum_{n=1}^{M} p_m^t q_m : \boldsymbol{q}\,(1-\alpha)\,S^0(y^0, z^0) + \alpha\,S^1(y^1, z^1)\} \tag{2}$$

Symbols ***p*** and ***q*** refer to price-quantity vectors. Feasibly set of quantities ***q*** is closed compact, because ***q*** is convex linear combination of the two quantities $\boldsymbol{q^0}$ and $\boldsymbol{q^1}$, *which are defined by statues of technologies in the two discrete periods - base 0 and current 1:* $S^0(u^0)$ and $S^1(u^1)$. Maximisation of output and minimisation of intermediary consumption - in monetary terms - is economically quite intuitive and it leads to maximisation of theoretical value added (*π*), building brick of GDP – by production approach, in the sense of equation (3):

$$\pi^t\,(p_y, p_x, z) = max_{y,x}\left\{\sum_{n=1}^{N} p_{yn}^t q_n - \sum_{n=1}^{M} p_{ym}^t q_m : \boldsymbol{q} \in S^t(z)\right\} \tag{3}$$

As regards expenditure side of GDP, Konüs index can be averaged in the same way as Fisher-Shell, by means of coefficient λ* ($0 \leq \lambda^* \leq 1$), according to equation (4), where symbol C^t stands for consumptions in the two discrete periods (base period 0 and current period 1):

$$P^t(p, \lambda^*) = C^t\,(p^1, \lambda^*) \,/\, C^t\,(p^0, \lambda^*) \tag{4}$$

Averaged theoretical Fisher-Shell indices of output Fisher-Shell {[6], 1972} intermediary consumption and value added can be constructed, equations (5)-(7), just as Konüs index in equation (4), in the following way, where all symbols have already been explained in equations (1)-(4). $S^t(z)$ is intermediary technology in discrete period t:

$$P\,(p^0, p^1, \alpha) = R(p^1, \alpha) \,/\, R(p^0, \alpha) \tag{5}$$

$$C\,(p^0, p^1, \alpha) = C(p^1, \alpha) \,/\, C(p^0, \alpha) \tag{6}$$

$$P^t(p, \alpha) = \pi^t\,(p^1, \alpha) \,/\, \pi^t\,(p^0, \alpha) \tag{7}$$

The derivations of all formulae form (1) to (7) are in IMF/.../World Banka {[8], chapter 17, pp. 435-462, 2004}. Equations (5)-(7) where both discrete periods, base 0 and current t, have been taken into account, form the basis for undoubtedly selection of symmetric T and F indices in constructing LMTF index. Fundamental feature of F index is that it is L and P bordered. That gives it advantage even against T index, but the latter is less restrictive because it allows increasing returns to scale what F index does not do. Diewert quadratic approximation lemma, primarily for the purpose of transparency, can be shown formally by the system of equations (8):

$$P_T(p^0, p^1; q^0, q^1;) = P^r\,(p^0, p^1; q^0, q^1;) = P^{s*}\,(p^0, p^1; q^0, q^1;)$$

$$\frac{\delta P_t(p^0,p^1,q^0,q^1)}{\delta p_i^t}=\frac{\delta P^r(p^0,p^1,q^0,q^1)}{\delta p_i^t}=\frac{\delta P^{s*}(p^0,p^1,q^0,q^1)}{\delta p_i^t} \quad \text{i =1, …, n i t =0,1}$$

$$\frac{\delta P_t(p^0,p^1,q^0,q^1)}{\delta q_i^t}=\frac{\delta P^r(p^0,p^1,q^0,q^1)}{\delta q_i^t}=\frac{\delta P^{s*}(p^0,p^1,q^0,q^1)}{\delta q_i^t} \quad \text{i =1,…,n i t =0,1} \qquad (8)$$

$$\frac{\delta P_t(p^0,p^1,q^0,q^1)}{\delta p_i^t\ \delta p_k^t}=\frac{\delta P^r(p^0,p^1,q^0,q^1)}{\delta p_i^t\ p_k^t}=\frac{\delta P^{s*}(p^0,p^1,q^0,q^1)}{\delta p_i^t\ p_k^t} \quad \text{i =1,…,n i t =0,1}$$

$$\frac{\delta P_t(p^0,p^1,q^0,q^1)}{\delta p_i^t\ \delta q_k^t}=\frac{\delta P^r(p^0,p^1,q^0,q^1)}{\delta p_i^t\ q_k^t}=\frac{\delta P^{s*}(p^0,p^1,q^0,q^1)}{\delta p_i^t\ q_k^t} \quad \text{i =1,…,n i t =0,1}$$

$$\frac{\delta P_t(p^0,p^1,q^0,q^1)}{\delta q_i^t\ \delta q_k^t}=\frac{\delta P^r(p^0,p^1,q^0,q^1)}{\delta q_i^t\ q_k^t}=\frac{\delta P^{s*}(p^0,p^1,q^0,q^1)}{\delta q_i^t\ q_k^t} \quad \text{i =1,…,n i t =0,1}$$

In the system of equations (8) p^t and q^t stand for the n-dimensional price and quantity vectors, subscripts i and k are for two different commodity baskets. Šutalo {Šutalo [13], 2012, pp. 33-39} made full citation of Diewert derivation {Diewert, [5], 2002} of theoretical and empirical flexible functional forms. This is general formula by which “superlative” index can be expressed, depending upon its order (r). This was done because authors in this paper once again try to argue why they chose T and F for construction of LMTF in the empirical part of the paper. Empirical flexible functional forms for “superlative” volume and price indices, which have been derived from their empirical counterparts, are shown in equations (9) and (10) {Šutalo [13], 2012, pp. 37 and 39}:

$$Q_r = \left[\sum_{n=1}^{N}\left(\frac{q_n^1}{q_n^0}\right)^{r/2} s_n^0\right]^{1/r}\left[\sum_{n=1}^{N}\left(\frac{q_n^1}{q_n^0}\right)^{-r/2} s_n^1\right]^{-1/r} \qquad (9)$$

$$P_r = \left[\sum_{n=1}^{N}\left(\frac{p_n^1}{p_n^0}\right)^{r/2} s_n^0\right]^{1/r}\left[\sum_{n=1}^{N}\left(\frac{p_n^1}{p_n^0}\right)^{-r/2} s_n^1\right]^{-1/r} \qquad (10)$$

Symbols p_n^t and q_n^t, in equations (9)-(10) are for prices and quantities of individual n^{th} commodity, in a pair of baskets, in the two discrete periods. F index is obtained by inserting 2 in the exponents of equations (9) and (10) and rearranging these equations by simple algebraic manipulations. Logarithmic T is of order 0 and its derivation is slightly more complicated {Diewert, [5], 2002, pp. 66-68}. LM index is, no doubt, central for the empirical part of this paper. Formula for this index is shown by Diewert {ILO ... World Bank, [7], 2004, pp. 327}, although it was originally developed by Lloyd {Lloyd, [10], 1975} and Moulton {Moluton, [12], 1999}. It is of the form shown in equation (13) {Šutalo [17], 2012, pp. 54}:

$$c(p) \equiv \alpha_0 \left(\sum_{i=1'}^{n} \alpha_i\, p_i^{1-\sigma}\right)^{1/(1-\sigma)} \quad \text{ako je } \sigma \neq 1 \qquad (11)$$

$$\ln c(p) \equiv \alpha_0 + \sum_{i=1}^{n} \alpha_i \ln p_i \quad ako\, je\ \sigma \equiv 1$$

Lloyd and Moulton developed formula for LM index, from equation (11), primarily in consumption context. In {Šutalo [13], 2012, pp. 54} extension of LM index formula has be made as CES production function correspondent. More deep insight into CES production function is {Varian [16], 1992, pp. 17-19} for the purpose of using it in GDP production-side price-volume decomposition. CES correspondent LM index formula is shown in equation (12):

$$\ln f(p) \equiv \alpha_0 \left(\sum_{i=1'}^{n} \alpha_i p_i^{1-\sigma}\right)^{1/(1-\sigma)} \quad \text{ako je } \sigma \neq 1 \qquad (12)$$

$$\ln f(p) \equiv \alpha_0 + \sum_{i=1}^{n} \alpha_i \ln p_i \quad ako\ je\ \sigma \equiv 1$$

Instead of unit cost aggregative function *c(p),* which is function in prices - p*,* in equation (11), the authors introduced unit output aggregative function *f(p)* in equation (12). σ in equations (11)-(12) stands for, from microeconomic theory well known, elasticity of substitution {look at Varian ([16], 1992}. For the purpose of constructing additive AGDP and QGDP in national accounts' practice, in {Šutalo [13], 2012, pp. 48-49}, relative additive F index weights for volume variant of this index (Q_F) are shown in equations (13) and (14):

$$Q_F - 1 = \sum_{n=1}^{N} \{(1/[1+Q_F])\, Q_F + (Q_F/[1+Q_F])\, Q_F w_n^1\}\{q_n^1 - q_n^0\} \qquad (13)$$

$$Q_{Fn} \equiv (1/[1+Q_F]) w_n^0 + (Q_F/[1+Q_F])\, Q_F w_n^1 \qquad (14)$$

Weights w_n^t, i = 0,1; are normalized prices calculated according to formula $w_n^t = p_n^t / (p^t q)^t$. Dutch statistician Van Ijzeren {Van Ijzeren [15], 2012, pp. 108-110} developed weights for F index p_n^* which give aggregative consistent GDP volumes (GDP in previous year constant prices) in absolute terms, in the sense of "Equ (15)":

$$Q(p^0,p^1,q^0,q^1) = \sum_{n=1}^{N} p_n^* q_n^1 / \sum_{n=1}^{N} p_n^* q_n^0 \qquad (15)$$

$Q(p^0,p^1,q^0,q^1)$ is "true" volume index with equal weights in both discrete periods: base period = 0 and current period = 1. Weights p_n^* are of the form shown in "Equ (16)":

$$p_n^* = (1/2)\, p_n^0 + (1/2)\, p_n^1 / P_F(p^0,p^1,q^0,q^1);\ n = 1,2,\ldots N. \qquad (16)$$

$P_F(p^0,p^1,q^0,q^1)$, in equation (16), is Fisher price index, whilst p_n^t refers to individual prices for n^th commodity in a aggregate. $\boldsymbol{p}^t$ and $\boldsymbol{q}^t$ are price and volume vectors which belong to aggregates expressed in value terms. Diewert {Diewert [5], 2002, pp. 75} showed that weights from "Equ (16)"uniquely belong to F index. In this paper the importance of the following two index theory fruits is pointed out in order to include T and F into LMTF index as well as not to include Walsh (W) index. It should be noted that trans-logarithmic functional form of more general T {Caves, Christensen, Diewert [1], 1982, pp. 1410} ruins down into F, under restrictive assumptions on substitution coefficients, in the two discrete periods (base 0 and current 1). Besides, indices F in total and T partially (under restrictive assumption that VAT tax rates are equal in base 0 and current 1 periods) - as GDP deflators - give the same values of GDP volumes on expenditure and production sides, when they are used for deflation {Šutalo [13], 2012, pp. 113-120}.

1.2 Index number theory – axiomatic and stochastic approach

The two axiomatic lists of desirable properties which have to be fulfilled by T and F indices, with 17 axioms for the former and 20 axioms for the latter, will be shortly exposed hereunder {look at Diewert W. E. and Nakamura A. O. [4], 1993, pp. 317-353) and IMF/.../World Bank [7], 2004, pp. 289-311}. These two lists unanimously candidate T and F deterministic indices (where their weights are not stochastic) as the second best solution according the index number theory. A short literature analysis was also carried out in {Šutalo [13], 2012, pp. 77-84} to demonstrate superiority of Törnqvist-Theil (TT) {Theil ([14], 1967, pp. 136-137)} weighted index (stochastic version of deterministic Törnqvist index) against fixed weights (1/n) indices. Among the researches Carli {Carli ([2], 1804, pp. 297-366)} and Jevons {see Jevons ([9], 1863, 1884, pp. 119-150)} are the most known.

2 LLOYD-MOULTON-TÖRNQVIST-FISHER INDEX AND FISHER INDEX SUPORTED BY LLOYD-MOULTON-TÖRNQVIST-FISHER COUNTERPART

2.1 Construction of Lloyd-Moulton-Törnqvist-Fisher index

The central point of this paper is construction of LMTF index, which measures GDP decomposition better than classic chain-linking methodology does. The complete estimation procedure has been carried out in the case study of Croatia. Original data sources used for LMTF calculation are Croatian annual and quarterly GDP data from q1 2000 to q4 2007 shown in data files: AGDP current prices, QGDP current prices, AGDP chain linked and QGDP chain linked. The four mentioned data files are shown in the most up-left corner in "Fig 1". The most demanding part of LMTF (I) calculation, the first variant of LMTF model, has been done by econometric Lloyd-Moulton (LM) estimation. The central point of this estimation was calculation of 28 elasticities of substitution σ_j^{LM}, one for each q1 2000 – q4 2007 quarter. In order to calculate these elasticities, QGDP relative price deflators (I_i) and relative QGDP shares s_i^j – at previous years prices – have to be calculated. Both of the two just mentioned sets of indicators consist of 1540 pairs (56 NACE classes => 56*(56-1)/2 = 1540 pairs) of relative I_i and s_i^j.

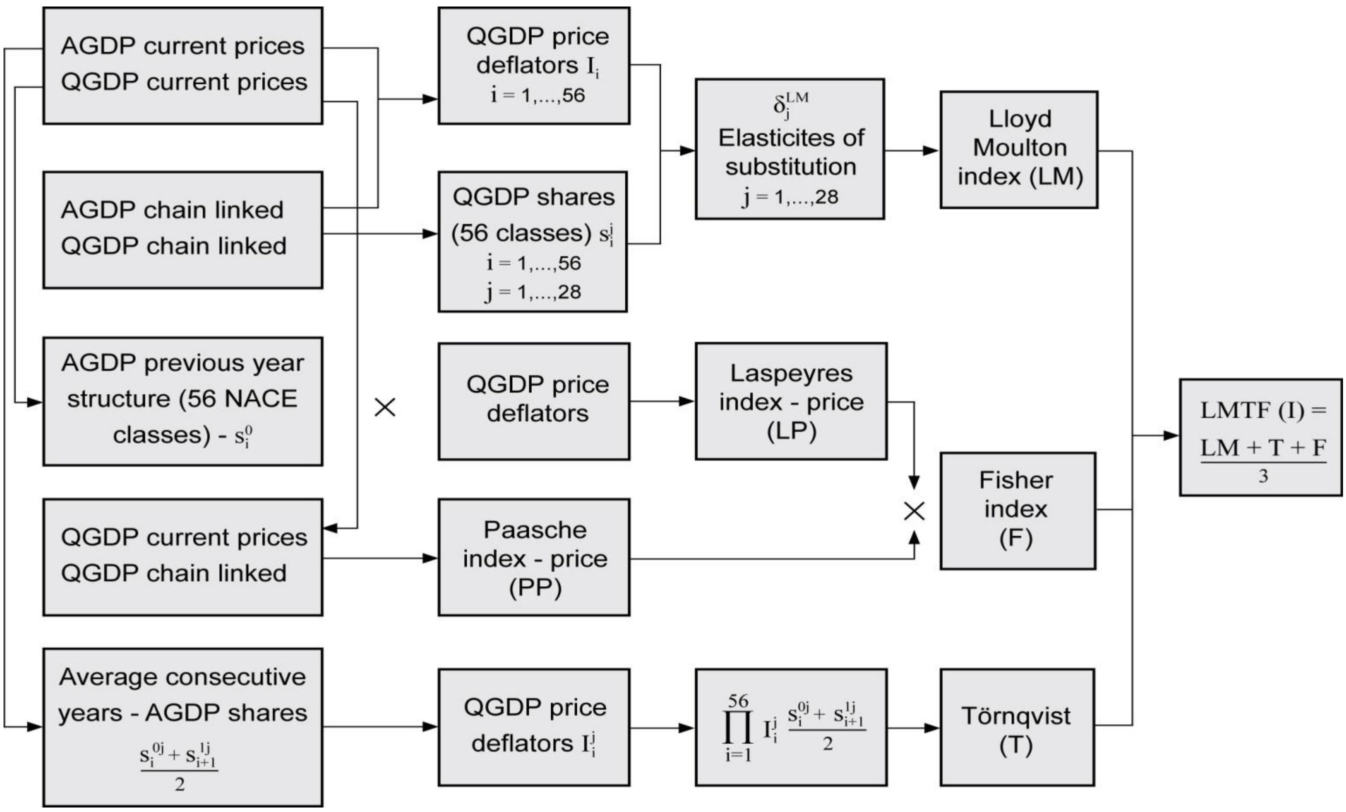

Fig.1. Scheme of the estimation procedure with the original data sources and intermediary tables for calculating Lloyd-Moulton-Törnqvist-Fisher index of type I.

Changes of GDP shares, among 1540 industries and between the two consecutive years (the same quarter of the current year through the same quarter of the previous year) and QGDP price deflators (just among 1540 industries) are in reverse order what is consistent with substitution behaviour of the Croatian producers. Namely, if GDP in industry *j* is getting "relative more expensive" compared to industry *i*, GDP share in i-th industry has to go down compared to industry *j*, and vice versa. Elasticities of substitution σ_j^{LM} are derived from econometric estimation of equation (17):

$$\ln\left[\frac{\left(\frac{s_i^{qt}}{s_j^{qt}}\right)}{\left(\frac{s_i^{qt-1}}{s_j^{qt-1}}\right)}\right] = \sigma^{LM} * \ln(P_j^{pt}/P_i^{pt}) + u_i, \; za \forall (i,j) \; i,j = 1...1540 \quad (17)$$

Parameter σ_j^{LM} is classic elasticity coefficient known from economic literature {Varian ([16], 1992, pp. 13-14)}. Looking at econometric estimation of σ_j^{LM} parameters, their significance and stability are of the crucial importance. Although data used in "Equ 17" are panel – especially on the right side of this equation, relative deflators are calculated for all 1540 among pairs of 56 NACE classes. The left side of "Equ 1" demonstrates panel data features. The data are among industries – cross section data - and in time – two consecutive years, which is characteristic of time series. Due to the time dimension of the data, the first passage through econometric software showed high positive autocorrelation demonstrated by very low Durbin-Watson (DW) statistics. In order to cure high positive autocorrelation, AR(1) model has been applied – using the first differences of the data. After the second passage though the econometric software - the following σ_j^{LM} estimates, shown in "Tab. 1", have been obtained:

Quarter (2)	Elasticities of substitution estimates (2)	t – statistics (3)	*p* - values t-stat. (4)	F – statistics (5)	*p* - values F stat. (6)*	DW (7)
q1 -2001.	0,0100	0,4247	$0,6711 \times 10^{0}$	0,1804	$0,6711 \times 10^{0}$	2,2679
q2 -2001.	0,2539	11,3435	$1,0645 \times 10^{-28}$	131,4093	$1,0645 \times 10^{-28}$	2,4105
q3 -2001.	0,2271	11,7450	$1,4000 \times 10^{-30}$	137,9443	$1,4000 \times 10^{-30}$	2,4354
q4 -2001.	0,1672	9,8756	$2,4146 \times 10^{-22}$	117,9162	2,4146 × 10-22	2,2693
q1 -2002.	0,6926	25,1191	$5,8000 \times 10^{-117}$	630,9682	$5,8000 \times 10^{-117}$	2,0321
q2 -2002.	0,7026	38,9803	$9,7000 \times 10^{-232}$	1519,4632	$9,7000 \times 10^{-232}$	2,2398
q3 -2002.	0,6775	38,6095	$1,4013 \times 10^{-228}$	1490,6898	$1,4013 \times 10^{-228}$	2,5561
q4 -2002.	0,5165	26,0736	$2,0657 \times 10^{-124}$	679,8304	$2,0657 \times 10^{-124}$	2,4679
q1 -2003.	0,8069	18,2341	$2,0857 \times 10^{-67}$	332,4825	$2,0857 \times 10^{-67}$	1,9642
q2 -2003.	0,9085	27,3031	$3,600 \times 10^{-134}$	745,4601	$3,600 \times 10^{-134}$	1,9660
q3 -2003.	1,0955	44,1131	$2,2357\times 10^{-235}$	1945,9640	$2,2357\times 10^{-235}$	2,1551
q4 -2003.	0,4506	6,6662	$3,6470 \times 10^{-11}$	44,4387	$3,6470 \times 10^{-11}$	1,0799
q1 -2004.	-0,0266	-0,7740	$0,4390 \times 10^{0}$	0,5991	$0,4390 \times 10^{0}$	2,12025
q2 -2004.	0,5100	17,1599	$1,5832 \times 10^{-60}$	294,4616	$1,5832 \times 10^{-60}$	2,1185
q3 -2004.	0,5717	23,3486	$1,8647 \times 10^{-103}$	545,1588	$1,8647 \times 10^{-103}$	2,1544
q4 -2004.	0,5384	26,5078	$7,7496 \times 10^{-128}$	702,6641	$7,7496 \times 10^{-128}$	2,2991
q1 -2005.	0,0370	1,5858	$0,1130 \times 10^{0}$	2,5146	$0,1130 \times 10^{0}$	2,29333
q2 -2005.	0,0956	10,7141	$6,9736 \times 10^{-26}$	114,7921	$6,9736 \times 10^{-26}$	0,6148
q3 -2005.	-0,2595	-11,1775	$6,0709 \times 10^{-28}$	124,9358	$6,0709 \times 10^{-28}$	2,3925
q4 -2005.	-0,2718	-13,7523	$1,1321 \times 10^{-40}$	189,1249	$1,1321 \times 10^{-40}$	2,5001
q1 -2006.	0,2618	14,2912	$1,3477 \times 10^{-43}$	204,2378	$1,3477 \times 10^{-43}$	2,4414
q2 -2006.	-0,1507	-5,5595	$3,1844 \times 10^{-8}$	30,9082	$3,1844 \times 10^{-8}$	2,4864
q3 -2006.	-0,2553	-8,3247	$1,8435 \times 10^{-16}$	69,3012	$1,8435 \times 10^{-16}$	2,5127
q4 -2006.	0,0377	2,4136	$0,0493\times 10^{0}$	204,2378	$0,0493\times 10^{0}$	2,4136
q1 -2007.	0,1794	14,9943	$1,5405 \times 10^{-47}$	224,8279	$1,5405 \times 10^{-47}$	2,4752
q2 -2007.	-0,0958	-3,5603	$0,0004 \times 10^{0}$	12,6758	$0,0004 \times 10^{0}$	2,6488
q3 -2007.	0,0316	1,5769	$0,1150 \times 10^{0}$	2,4865	0,1150 × 100	2,2983
q4 -2007.	-0,0586	-3,7962	$0,0002 \times 10^{0}$	14,4113	$0,0002 \times 10^{0}$	2,2954

Table1. Estimates of 28 elasticities of substitution, among 1539 industries' pairs by AR(1) transformation - data base of Croatian quarterly GDP - period from q2001 to q4 2007

As there is (on the right side of "Equ 1") only one exogenous variable, F statistic is t statistic squared (i.e. $F = t^2$). So, both of these two statistics indicate significance of σ_j^{LM} in the same way.

Situation regarding significance of σ_j^{LM} , after AR(1) transformation was applied, is the following: the four yellow highlighted rows indicate insignificant elasticities, while the two of them (q1 2001 and q4 2004) are absolutely insignificant and the other two are significant at 11% significance levels (these two quarters are q1 2005 and q3 2007). Elasticities in 18 quarters (white rows) are highly significant and positive, while in six quarters (gray rows) elasticities are highly significant but negative. It is important to notice that in all 28 quarters prior-expected positive substitution prevails. Average elasticity of substitution in all 28 quarters, taking into account minus and plus signs, amounts to 0,2734. Taking into account absolute values of all 28 elasticities, a slightly higher value has been gotten (0,3532) because six negative elasticities, indicating complementary - instead of substitutive relations, possess not too big absolute values. In order to get LM price and volume indices, σ_j^{LM} has to be inserted into exponent of empirical LM indices derived from their theoretical counterparts. It is done - according to equation (18) and (19) {Šutalo [13], 2012, pp. 55}:

$$P_{LM}(p^0, p^1, q^0, q^1) \equiv \left\{ \sum_{i=1}^{n} s_i^0 \left(\frac{p_i^1}{p_i^0} \right)^{1-\sigma} \right\}^{(1/1-\sigma)} \quad (18)$$

$$Q_{LM}(p^0, p^1, q^0, q^1) \equiv \left\{ \sum_{i=1}^{n} s_i^0 \left(\frac{q_i^1}{q_i^0} \right)^{1-\sigma} \right\}^{(1/1-\sigma)} \quad (19).$$

P_{LM} and Q_{LM} , in equations (18)-(19) are price-volume Lloyd-Moulton indices s_i^0 in equations (18)-(19), are volume shares of commodity *i* in the overall production-consumption aggregates. All other symbols have been already known from the equations preceding equations (18)-(19). Once LM index had been prepared, it was used as the first (and the most important one) component of LMTF (I). How the second component of LMTF index, Fisher (F) index, was calculated is shown in the middle part of "Fig. 1". The first component of F index, price Laspeyres (L), was calculated by weighting 56 QGDP price deflators using previous years AGDP shares as weights. The second component of F index, price Paasche (P), was calculated by simple dividing 56 QGDP current price GDPs through 56 QGDP chain linked (previous year prices) data. The last holds for the fact implicit GDP deflators, where current price values being divided by volume (aggregate at previous year prices), are of Paasche type in mathematical sense {look at Šutalo [13], 2012, pp. 89-90}. Multiplying price Laspeyres (LP) and price Paasche (PP), and putting this product under square root, price Fischer (F) is gotten. This is in accordance with well known formula for F index where the last one is geometric average of the previous two. The down-most line of "Fig. 1" shows sequential steps how Törnqvist index (T) has been calculated. Namely, once average consecutive years – AGDP shares had been prepared, they were used for weighting QGDP price deflators (for all 28 quarters) using geometric mean formula as a basis (look at the second last text box in the last row of the "Fig. 1"). Once LM price index was calculated, according to equation (18), what is not straightforward at all - due to the implicit form of equation (18), σ (elasticity of substitution) appears firstly within curly brackets as an exponent above 56 deflators - and, once again, above curly brackets. LM price index was used as component in simple averaging, together with F and T, to arrive at LMTF (I).

"Fig. 2" offers a short description of the second variant of LMTF (LMTF (II)) construction. The whole procedure is continuation of the process described in "Fig. 1". It starts from T and F indices, the second right-most column in "Fig. 1". Once T and F were calculated, formula (18) has been used to determine which values of σ_j^T and σ_j^F (i.e. T and F correspondent elasticities) parameters fit to the advance determined T and F. Calculation of σ_j^T and σ_j^F is not straightforward at all, due to the implicit form of equation (18). But this calculation requires numeric (iterative) procedure. Thanks to the fact iterative procedures - like this one - which is available in numerous software (even in Excel), calculation of adequate elasticities of substitution, is pretty simple. Once σ_j^T and σ_j^F were calculated by this iterative procedure, and after σ_j^{LM} (i.e. LM correspondent elasticity) was undertaken from econometrics' module, they were averaged as a simple mean (look at the second right-most box in "Fig. 2"). In order to arrive at LMTF (II) index, recursive procedure was followed. Average elasticities

$\overline{\sigma_j}$ were inserted backwards into equation LMTF(II) (look at the right-most box in "Fig. 2"), for all 28 quarters (q1 2000-q4 2007).

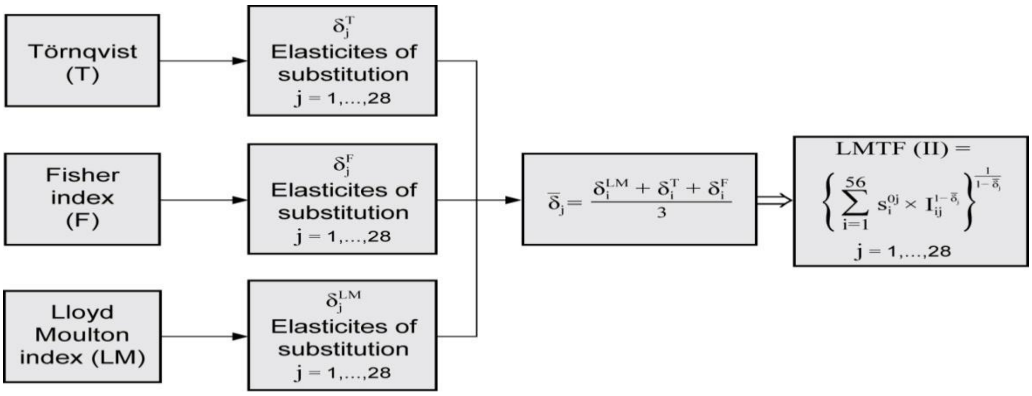

Fig. 2. Törnqvist, Fisher and Lloyd-Moulton indices as starting points for calculating Lloyd-Moulton-Törnqvist-Fisher index of type II.

Using LMTF (I) from "Fig 1" and LMTF (II) from "Fig 2" for deflation of nominal GDPs (annual and quarterly), GDP rates of changes shown in "Tab 2" can be estimated. For the key assertion of this paper, LMTF index factorises GDP better than classic chain linking, columns (3) and (4) are of special interest. Namely, differences in GDPs' rates of change, coming from the classic approach and the new approach suggested by this paper (LMTF index), are shown. These two columns in "Tab. 2" show significant differences in the following quarter, for LMTF (II) and LMTF(I) respectively. The biggest positive differences are in q3 2003 and q4 2006 and the biggest negative differences belong to q3 2002 and q2 2001.

Ordinal number of the quarter (1)	**Quarter (2)**	**Differences: LMTF (II) against classic calculation (CBS) (3)**	**Differences: LMTF (I) against classic calculation (CBS) (4)**
1	q1 -2000.	-	-
2	q2 -2000.	-	-
3	q3 -2000.	-	-
4	q4 -2000.	-	-
	2000./1999.	**-**	
5	q1 -2001.	0,1356	0,1347
6	q2 -2001.	-0,2242	**-0,2240**
7	q3 -2001.	0,1991	0,1980
8	q4 -2001.	-0,1180	-0,1184
	2001./2000.	**-0,0010**	**-0,0015**
9	q1 -2002.	0,2526	0,2568
10	q2 -2002.	0,3721	0,3685
11	q3 -2002.	**-0,4711**	-0,1723
12	q4 -2002.	-0,0299	-0,0307
	2002./2001.	**0,0217**	**0,1006**
13	q1 -2003.	0,2738	0,3432
14	q2 -2003.	0,1299	0,2112
15	q3 -2003.	**0,5773**	0,3643
16	**q4 -2003.**	**0,2459**	**0,3256**
	2003./2002.	0,3131	0,3128
17	q1 -2004.	0,0562	0,0525
18	q2 -2004.	0,2171	0,2148
19	q3 -2004.	0,0959	0,0877
20	q4 -2004.	-0,2106	-0,2135
	2004./2003.	**0,0417**	**0,0374**
21	q1 -2005.	0,1039	0,1118
22	q2 -2005.	-0,1019	-0,1058

23	q3 -2005.	0,0544	0,0533
24	q4 -2005.	0,3172	0,3169
	2005./2004.	**0,0901**	**0,0907**
25	q1 -2006.	-0,0209	-0,0221
26	q2 -2006.	0,0521	0,0539
27	q3 -2006.	0,0702	0,0665
28	q4 -2006.	0,3863	**0,3857**
	2006./2005.	**0,1235**	**0,1225**
29	q1 -2007.	-0,0260	-0,0251
30	q2 -2007.	0,2464	0,2455
31	q3 -2007.	0,1359	0,1402
32	q4 -2007.	0,1519	0,1527
	2007./2006.	**0,1300**	**0,1314**

Table 2. Differences in Quarterly Gross domestic (QGDP) product rates of change calculated by classic Central Bureau of statistic (CBS) approach and alternative rates of change of QGDP calculated using Lloyd-Moulton-Törnqvist-Fisher index (LMTF) index (variants I and II).

Usage of LMTF index is recommended anyway, irrespectively of its variants, because it:

a) measures substitution in the better way than classic chain linking methodology (due to econometric estimation of this phenomenon) and

b) LMTF allows (due to inclusion of T into LMTF structure) increasing returns to scale.

2.2 Fisher index supported by Lloyd-Moulton-Törnqvist-Fisher counterpart

Beside the prime goal of the paper-improvement of GDP price-volume decomposition, the second - not less important - goal has been resolving of additivity problem (see “Fig. 3”). This is not as important for the quality of GDP compilation as it is for the quality of GDP publication (dissemination). Namely, users like to see GDP components (in volume terms) additive into aggregate.

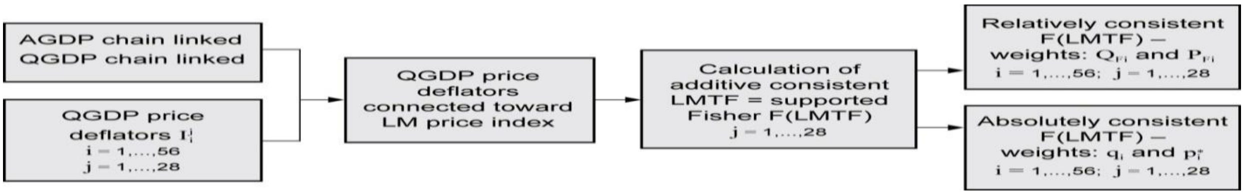

Fig.3. Construction scheme of Fisher index supported by Lloyd-Moulton- Törnqvist-Fisher counterpart

Following procedure announced in “Fig. 3” Fisher index supported by Lloyd-Moulton relative additive consistent decomposition of quarterly GDP in volume terms is shown in “Tab. 3” (2000 is referent year).

Ordinal number of the quarter (1)	Quarter (2)	F volume (from rel. additive decomposition) (3)	F volume (directly calculated) (4)	Differ. (5) = (3) – (4)	Escalating factors (6) = (4)/(3)
1	q1 -2001.	98,5122	98,5222	-0,0100	**1,0001**
2	q2 -2001.	104,2357	104,5490	-0,3132	**1,0030**
3	q3 -2001.	110,0894	110,1655	-0,0761	**1,0007**
4	q4 -2001.	103,8990	104,1548	-0,2559	**1,0025**
5	q1 -2002.	103,1878	103,2730	-0,0853	**1,0008**
6	q2 -2002.	108,9976	109,3129	-0,3153	**1,0029**
7	q3 -2002.	117,6408	117,8633	-0,2225	**1,0019**
8	q4 -2002.	109,8143	110,3038	-0,4895	**1,0045**
9	q1 -2003.	109,0044	109,0821	-0,0777	**1,0007**
10	q2 -2003.	115,6576	116,3967	-0,7391	**1,0064**
11	q3 -2003.	124,0039	124,4059	-0,4021	**1,0032**
12	q4 -2003.	114,0826	114,7483	-0,6657	**1,0058**
13	q1 -2004.	113,8005	114,0205	-0,2200	**1,0019**
14	q2 -2004.	120,0054	121,0910	-1,0856	**1,0090**

Ordinal number of the quarter (1)	Quarter (2)	F volume (from rel. additive decomposition) (3)	F volume (directly calculated) (4)	Differ. (5) = (3) – (4)	Escalating factors (6) = (4)/(3)
15	q3 -2004.	127,9108	129,0358	-1,1251	**1,0088**
16	q4 -2004.	117,2792	118,8590	-1,5798	**1,0135**
17	q1 -2005.	116,3581	116,7651	-0,4070	**1,0035**
18	q2 -2005.	125,5378	126,9434	-1,4056	**1,0112**
19	q3 -2005.	133,9299	135,2211	-1,2912	**1,0096**
20	q4 -2005.	122,1814	123,8983	-1,7168	**1,0141**
21	q1 -2006.	122,7044	123,1056	-0,4012	**1,0033**
22	q2 -2006.	130,0313	131,5019	-1,4706	**1,0113**
23	q3 -2006.	139,9048	141,3740	-1,4692	**1,0105**
24	q4 -2006.	128,1195	129,7245	-1,6050	**1,0125**
25	q1 -2007.	130,6500	131,3048	-0,6548	**1,0050**
26	q2 -2007.	137,8283	139,3924	-1,5640	**1,0113**
27	q3 -2007.	146,7078	148,4296	-1,7218	**1,0117**
28	q4 -2007.	133,0943	134,8577	-1,7634	**1,0132**

Table 3. Relative additive LMTF supported Fisher index – 2000 is referent year

Fisher (F) index supported by Lloyd-Moulton-Törnqvist-Fisher (LMTF) is F index, rescaled on 2000 referent year, and in “Tab. 3” it is calculated in two ways:

a) applying relative additive weights (weights defined in “Equ 14”) and

b) directly, applying classic F formula.

The authors suggest usage of these weights (from “Equ 14”) to Croatian Central bureau of statistics – although the same can be adopted by any other national statistical agency. Namely, “Equ (13)” and “Equ (14)” give the scheme for averaging absolute changes of current quarter GDP versus its value in referent 2000 prices by all 56 NACE classes. In other words 56 $\{q_n^1 - q_n^0\}$ have been averaged, according to “Equ (13)”). So, if all current quarter GDPs are put onto 2000 referent year prices by F commensurate to LM, index (q_n^1) in “Equ (13)”, and ¼ of nominal annual GDP from referent 2000 is subtracted from q_n^1, one arrives at the difference $\{q_n^1 - q_n^0\}$. Averaging of these deltas by weights from “Equ (14)” will give Fisher rate of change (F supported by LMTF). From the practical point of view, i.e. additive consistent dissemination of AGDPs and QGDPs, absolute (Van-Ijzeren) decomposition looks like much better, and it is shown in “Tab.4”.

Ordinal number of the quarter (1)	Quarter (2)	F volume (from abs. additive decomposition) (3)	F volume (directly calculated) (4)	Differ. (5) = (3) – (4)	Escalating factors (6) = (4)/(3)
1	q1 -2001.	98,4972	98,5222	-0,0251	**1,0003**
2	q2 -2001.	104,8750	104,5490	0,3261	**0,9969**
3	q3 -2001.	110,1251	110,1655	-0,0404	**1,0004**
4	q4 -2001.	104,3373	104,1548	0,1825	**0,9983**
5	q1 -2002.	103,2384	103,2730	-0,0347	**1,0003**
6	q2 -2002.	109,5594	109,3129	0,2464	**0,9978**
7	q3 -2002.	117,9522	117,8633	0,0890	**0,9992**
8	q4 -2002.	110,7391	110,3038	0,4352	**0,9961**
9	q1 -2003.	109,0465	109,0821	-0,0357	**1,0003**
10	q2 -2003.	117,1334	116,3967	0,7367	**0,9937**
11	q3 -2003.	124,6909	124,4059	0,2850	**0,9977**
12	q4 -2003.	115,3923	114,7483	0,6439	**0,9944**

Ordinal number of the quarter (1)	Quarter (2)	F volume (from abs. additive decomposition) (3)	F volume (directly calculated) (4)	Differ. (5) = (3) – (4)	Escalating factors (6) = (4)/(3)
13	q1 -2004.	114,1170	114,0205	0,0965	**0,9992**
14	q2 -2004.	122,2446	121,0910	1,1537	**0,9906**
15	q3 -2004.	130,2187	129,0358	1,1828	**0,9909**
16	q4 -2004.	120,5753	118,8590	1,7163	**0,9858**
17	q1 -2005.	117,0682	116,7651	0,3030	**0,9974**
18	q2 -2005.	128,5174	126,9434	1,5740	**0,9878**
19	q3 -2005.	136,6503	135,2211	1,4292	**0,9895**
20	q4 -2005.	125,8197	123,8983	1,9214	**0,9847**
21	q1 -2006.	123,3756	123,1056	0,2699	**0,9978**
22	q2 -2006.	133,1738	131,5019	1,6719	**0,9874**
23	q3 -2006.	143,0578	141,3740	1,6838	**0,9882**
24	q4 -2006.	131,5383	129,7245	1,8138	**0,9862**
25	q1 -2007.	131,8471	131,3048	0,5424	**0,9959**
26	q2 -2007.	141,2395	139,3924	1,8471	**0,9869**
27	q3 -2007.	150,5377	148,4296	2,1081	**0,9860**
28	q4 -2007.	136,9277	134,8577	2,0700	**0,9849**

Table 4. Absolutely additive LMTF supported Fisher index – 2000 is referent year

In “Tab. 4” absolute additive decomposition of F index supported by LMTF is shown. Derivation of this type of index (column (3) in “Tab. 4” is carried out using weights from “Equ (16)”. This form of GDP dissemination, using “Equ (15)”, is much better for dissemination purposes. Namely, current quarter aggregate GDP volumes in absolute terms (denominator in “Equ (15)”) is absolutely aggregative consistent (i.e. GDP levels are aggregative). Likewise referent year GDP level – denominator in “Equ (15)” is absolutely aggregative consistent. This is clearly demonstrated by Case study of Croatia with 2000 as referent year. Aggregative consistent GDP volume from current quarter (nominator in “Equ (15)”) divided by consistent GDP volume from referent 2000 year (denominator in “Equ (15)”) gives “ideal” F volume index supported by LMTF. It is undoubtedly empirical improvement of ordinary applied chain-linking methodology.

3 CONCLUSION

The main goal of this paper has been to establish a new methodological approach to upgrading the statement of Gross domestic product (GDP) growth rates and implicit GDP deflators – on annual and quarterly bases. The traditional methodological approach in the practice of National statistical agencies around the world is the chain-linking methodology. The main mathematical apparatus of the chain-linking methodology are the two indices. The first one, Laypeyres index with fixed base substantially overestimates the second, Paasche index, which is the most appropriate GDP deflator due to statistical (Cauchy theorem) and economic (substitution-transformation effect) reasons. By means of chain linking, index number drift has been resolved partially in the sense of the second best solution. But index number mathematics provides a solution. By it's theoretical considerations Törnqvist and Fisher indices have been chosen among so called “superlative indices” as superior ones for the GDP compilation. According econometric estimations Lloyd-Moulton index has been also calculated as the best estimator of elasticity of substitution. Putting together Lloyd-Moulton with Törnqvist and Fisher indices, authors have constructed Lloyd-Moulton-Törnqvist-Fisher (LMTF) model. LMTF model improves GDP price-volume decomposition due to more precise substitution measurement. Fisher index supported by LMTF model has been also built and it resolves the problem of additive (absolute and relative) inconsistency in GDP data. The whole estimation procedure has been implemented on the case study of Croatia. The data base dealing with Croatian Quarterly GDP data has related to the period from q1 2000 to q4 2007. Thanks to the approach proposed in this paper, ex-post smoothing of the preliminary raw-data driven by original (price and volume) indicators preserves indicators content of GDP data but improve “maturity” of GDP data. An integral part of the survey are testing results which prove that Fisher index supported by LMTF model can be considered as "ideal" in the practical applications. Namely, the new methodological approach proposed in this paper has at least three advantages : a) better decomposes “mature” GDP data on price and volume, b) assures additive consistent GDPs for publication and c) preserves (by means of F supported by LMTF) product test identity (value = volume times price). This is the reason to choose it.

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