

Combine Laplace Transform and Variational Iteration Method to Solve Convolution Differential Equations

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Abstract

In this work we combine Laplace transform and modified variational iteration method to solve new type of differential equation called convolution differential equations, it is possible to find the exact solutions or better approximate solutions of these equations. In this method, a correction functional is constructed by a general Lagrange multiplier, which can be identified via variational theory. This method is used for solving a convolution differential equation with given initial conditions. The solutions obtained by this method show the accuracy and efficiency of the method.

Keywords - Modified Variational Iteration Method, Convolution Differential Equations, Lagrange Multiplier and Laplace Transform.

1. INTRODUCTION

It is well known that there are many nonlinear differential equations which are used in the study of several fields for example physics, mechanics, etc. The solutions of these equations can give more understanding of the described process. But because of the complexity of the nonlinear differential equations and the limitations of mathematical methods, it is difficult to obtain the exact solutions for these problems. Thus, this complexity hinders further applications of nonlinear differential equations [1, 2]. A broad class of analytical and numerical methods were used to handle these problems such as Backlund transformation [3], Hirota's bilinear method [4, 5], Darboux transformation [6], Symmetry method [7], the inverse scattering transformation [8], the tanh method [9, 10], the Adomian decomposition method [11, 12], the improved Adomian decomposition method [13], the exp-function method [14] and other asymptotic methods [15] for strongly nonlinear equations. In 1978, Inokuti et al. [16] proposed a general use of Lagrange multiplier to solve nonlinear problems, which was intended to solve problems in quantum mechanics. Subsequently, in 1999, the variational iteration method (VIM) was first proposed by Ji-Huan He [17, 18]. The idea of the VIM is to construct an iteration method based on a correction functional that includes a generalized Lagrange multiplier. The value of the multiplier is chosen using variational theory so that each iteration improves the accuracy of the solution.

In this paper, we have applied the modified variational iteration method (VIM) and Laplace transform to solve a new type of equations called convolution differential equations.

1.1 Definitions

Let $f(x)$, $g(x)$ be integrable functions, then the convolution of $f(x)$, $g(x)$ is defined

$$\text{as } f(x) * g(x) = \int_0^x f(x-t)g(t)dt$$

And the Laplace transform is defined as

$$L[f(x)] = F(s) = \int_0^{\infty} e^{-sx} f(x) dx$$

Where $x > 0$ and s is complex value.

And further the Laplace transform of first and second derivatives are given by

$$(i) L[f'(x)] = sL[f(x)] - f(0) \quad (ii) L[f''(x)] = s^2L[f(x)] - sf'(0) - f''(0)$$

More generally

$$L[f^{(n)}(x)] = s^n L[f(x)] - s^{n-1}f'(0) - s^{n-2}f''(0) - \dots - sf^{(n-2)}(0) - f^{(n-1)}(0)$$

Theorem (Convolution Theorem)

If $L[f(x)] = F(s)$, $L[g(x)] = G(s)$, then:

$$L[f(x)*g(x)] = L[f(x)]L[g(x)] = F(s)G(s) , \text{ or equivalently,}$$

$$L^{-1}[F(s)G(s)] = f(x)*g(x)$$

2. Variational Iteration Method (VIM)

Consider the differential equation

$$L[y(x)] + R[y(x)] + N[y(x)] + N^*[y(x)] = 0 \tag{1}$$

With the initial conditions

$$y(0) = h(x) \quad , \quad y'(0) = k(x) \tag{2}$$

Where L is a linear second order operator, R is a linear operator less than L , N is the nonlinear operator, and $N^*[y(x)]$ is the nonlinear convolution term which is definite by

$$N^*[y(x)] = f(y, y', y'', \dots, y^{(n)}) * g(y, y', y'', \dots, y^{(n)})$$

According to variational iteration method, we can construct a correction functional as follows

$$y_{n+1}(x) = y_n(x) + \int_0^x \lambda(\zeta) [Ly_n(\zeta) + R\tilde{y}_n(\zeta) + N\tilde{y}_n(\zeta) + N^*\tilde{y}_n(\zeta)] d\zeta \tag{3}$$

$R\tilde{y}_n(\zeta), N\tilde{y}_n(\zeta)$ and $N^*\tilde{y}_n(\zeta)$ are considered as restricted variations, i.e.

$$\delta R\tilde{y}_n = 0 \quad , \quad \delta N\tilde{y}_n = 0 \quad \text{and} \quad \delta N^*\tilde{y}_n = 0 \quad , \quad \lambda = -1$$

Then the variational iteration formula can be obtained as

$$y_{n+1}(x) = y_n(x) - \int_0^x [Ly_n(\zeta) + Ry_n(\zeta) + Ny_n(\zeta) + N^*y_n(\zeta)] d\zeta \tag{4}$$

Eq (4) can be solved iteratively using $y_0(x)$ as the initial approximation.

Then the solution is $y(x) = \lim_{n \rightarrow \infty} y_n(x)$

3. Modified Variational Iteration Laplace Transform Method (MVILTM)

Consider the nonlinear convolution ordinary differential equation

$$L[y(x)] + R[y(x)] + N[y(x)] + N^*[y(x)] = 0 \quad (5)$$

With the initial conditions

$$y(0) = h(x) \quad , \quad y'(0) = k(x) \quad (6)$$

Where L is a linear operator, R is a linear operator less than L and N is the nonlinear operator, and $N^*[y(x)] = f(y, y', y'', \dots, y^{(n)}) * g(y, y', y'', \dots, y^{(n)})$ is the nonlinear convolution term.

In this work we assume that $L = \frac{d^2}{dx^2}$

Take Laplace transform on both sides of eq (5), to find

$$\ell[Ly(x)] + \ell[Ry(x)] + \ell[Ny(x)] + \ell[N^*y(x)] = 0 \quad (7)$$

$$s^2 \ell y - sy(0) - y'(0) = -\ell\{Ry(x) + Ny(x) + N^*y(x)\} \quad (8)$$

By using the initial conditions and taking the inverse Laplace transform we have

$$y(x) = p(x) - \ell^{-1} \left[\frac{1}{s^2} \ell\{Ry(x) + Ny(x) + N^*y(x)\} \right] \quad (9)$$

Where $p(x)$ represents the terms arising from the source term and the prescribed initial conditions.

Now the first derivative of eq (9) is given by

$$\frac{dy(x)}{dx} = \frac{dp(x)}{dx} - \frac{d}{dx} \ell^{-1} \left[\frac{1}{s^2} \ell\{Ry(x) + Ny(x) + N^*y(x)\} \right] \quad (10)$$

By the correction function of the irrational method we have

$$y_{n+1}(x) = y_n(x) - \int_0^x \left\{ (y_n(\zeta))_\zeta - \frac{d}{d\zeta} p(\zeta) - \frac{d}{d\zeta} \ell^{-1} \left[\frac{1}{s^2} \ell\{Ry_n(\zeta) + Ny_n(\zeta) + N^*y_n(\zeta)\} \right] \right\} d\zeta$$

Then the new correction function (new modified VIM) is given by

$$y_{n+1}(x) = y_n(x) + \ell^{-1} \left[\frac{1}{s^2} \ell\{Ry_n(x) + Ny_n(x) + N^*y_n(x)\} \right], \quad n \geq 0 \quad (11)$$

In the last we find the solution in the form $y(x) = \lim_{n \rightarrow \infty} y_n(x)$, if inverse Laplace transform exist.

In particular consider the nonlinear ordinary differential equations with convolution terms

$$1- \quad y''(x) - 2 + 2y' * y'' - y' * (y'')^2 = 0 \quad , \quad y(0) = y'(0) = 0 \quad (12)$$

Take Laplace transform of eq (12), and making use of initial conditions, we have

$$s^2 \ell y(x) - \frac{2}{s} = \ell[y' * (y'')^2 - 2y' * y'']$$

The inverse Laplace transform implies that

$$y(x) = x^2 + \ell^{-1} \left\{ \frac{1}{s^2} \ell[y' * (y'')^2 - 2y' * y''] \right\}$$

By using the new modified (eq (11)), we have the new correction function

$$y_{n+1}(x) = y_n(x) + \ell^{-1} \left\{ \frac{1}{s^2} \ell \left[y_n' * (y_n'')^2 - 2y_n' * y_n'' \right] \right\}$$

Or

$$y_{n+1}(x) = y_n(x) + \ell^{-1} \left\{ \frac{1}{s^2} \left[\ell(y_n') \ell(y_n'')^2 - 2\ell(y_n') \ell(y_n'') \right] \right\} \tag{13}$$

Then we have

$$\begin{aligned} y_0(x) &= x^2 \\ y_1(x) &= x^2 + \ell^{-1} \left\{ \frac{1}{s^2} \left[\ell(4) \ell(2x) - 2\ell(2x) \ell(2) \right] \right\} = x^2 \\ y_2(x) &= x^2, \quad y_3(x) = x^2, \quad \dots, \quad y_n(x) = x^2 \end{aligned}$$

This means that $y_0(x) = y_1(x) = y_2(x) = \dots = y_n(x) = x^2$

Then the exact solution of eq (12) is $y(x) = x^2$

$$2- y' - (y')^2 - 2x + y' * (y'')^2 = 0, \quad y(0) = 1 \tag{14}$$

Take Laplace transform of eq (14), and use the initial condition, we obtain

$$s \ell y - 1 - \frac{2}{s^2} = \ell \left[(y')^2 - y' * (y'')^2 \right]$$

Take the inverse Laplace transform to obtain

$$y(x) = 1 + x^2 + \ell^{-1} \left\{ \frac{1}{s} \ell \left[(y')^2 - y' * (y'')^2 \right] \right\}$$

Using eq (11) to find the new correction function in the form

$$y_{n+1}(x) = y_n(x) + \ell^{-1} \left\{ \frac{1}{s} \ell \left[(y_n')^2 - y_n' * (y_n'')^2 \right] \right\}$$

Or

$$y_{n+1}(x) = y_n(x) + \ell^{-1} \left\{ \frac{1}{s} \left[\ell \left[(y_n')^2 \right] - \ell \left[y_n' \right] \ell \left[(y_n'')^2 \right] \right] \right\} \tag{15}$$

Then we have

$$\begin{aligned} y_0(x) &= 1 + x^2 \\ y_1(x) &= 1 + x^2 + \ell^{-1} \frac{1}{s} \left\{ \ell(4x^2) - \ell(2x) \ell(4) \right\} = \\ &= 1 + x^2 + \ell^{-1} \frac{1}{s} \left\{ \frac{8}{s^3} - \left(\frac{2}{s^2} \right) \left(\frac{4}{s} \right) \right\} = 1 + x^2 \\ &\dots\dots\dots \\ y_0(x) &= y_1(x) = y_2(x) = \dots = y_n(x) = 1 + x^2 \end{aligned}$$

Then the exact solution of eq (14) is $y(x) = 1 + x^2$

4. Conclusion

In this paper we introduce new differential equations called nonlinear convolution ordinary differential equations, and solve them by combine Laplace transform and new modified variational iteration method. The method is applied in a direct way without using linearization. This method is successfully implemented by using the initial conditions.

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