Hosoya Polynomials of Coalescence and Bridges Coalescence Graphs

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Abstract

In this paper , we find Hosoya polynomial and Wiener index of n-connected graphs by the vertices identified and edges introducing with isolated vertex. Also , we find Hosoya polynomial , Wiener index , and average distance for some compound graphs from special graphs

Keyword : distance, Hosoya Polynomial, Wiener index, average distance.

1.Introduction:

Let G be a connected graph of order p and size q. The **distance** between two vertices u and v in G is the minimum of the lengths of *u*-*v* paths in G ;it is denoted by $d(u,v)$.

The **eccentricity** $e(v)$ of a vertex v in G is defined by :

$$e(v)=max\{d(u,v):u\in V(G)\},$$

where $V(G)$ is the vertex set of G. The **diameter of G** $diamG$, or δ ,is defined by:

$$diamG=max\{e(v):v\in V(G)\}=max\{d(u,v):u,v\in V(G)\}.$$

Let $d(G,k)$ be the number of pairs of vertices in G that are distance k apart ,$0\le k\le\delta$, then the Hosoya polynomial of a connected graph G is defined by [8]

$$H(G;x)=\sum_{k=0}^{\delta}d(G,k)x^k \ .$$

Let $d(v,G,k)$ be the number of vertices in G, that are at distance k from vertex v, then the Hosoya polynomial of a vertex v is defined by [8]

$$H(v,G;x)=\sum_{k\ge 0}d(v,G,k)x^k \ .$$

Observe that $H(v,G;0)=1$, $H(v,G;1)=p$, $H(G;0)=p$, and $H(G;1)=q$.

The concept of distance based polynomial was introduced in 1988 by H. Hosoya [6] Hosoya's paper, which seems to be the only published work concerning this issue, reported a limited number of results on Hosoya polynomials. In 1993, Gutman [5] defined vertex identified and edge introducing graphs, constructed from two vertex disjoint connected graphs as given next:

Definition:[5] Let G_1 and G_2 be any two connected graphs with disjoint vertex sets. Let u and v be any two vertices of G_1 and G_2 respectively, then the vertex identified graph $G_1\cdot G_2$ is obtained from G_1 and G_2 by identifying the vertices u and v, the edge introduced graph $G_1:G_2$ is the graph obtained from G_1 and G_2 by introducing a new edge joining the vertices u and v . Gutman obtained $H(G_1\cdot G_2;x)$ and $H(G_1:G_2;x)$ as given next.

Gutman's Theorem:[5]

(a). $H(G_1\cdot G_2;x)=H(G_1;x)+H(G_2;x)+H(u,G_1;x)H(v,G_2;x)$
$-H(u,G_1;x)-H(v,G_2;x)$.

(b). $H(G_1:G_2;x)=H(G_1;x)+H(G_2;x)+x\,H(u,G_1;x)H(v,G_2;x)$. #

Our terminology and notations will be as given in the references [3,4].

2. Hosoya Polynomials of Coalescence Graphs:

Let G_i be a connected graph of order p_i and size q_i with $u_i \in G_i$, $i = 1,2,\ldots,n$, define the **coalescence graph** $\prod_{i=1}^{n}(u_i, G_i, .)$ is the graph obtained from $G_1, G_2, \ldots, G_n$ by identifying the vertices $u_1, u_2, \ldots, u_n$. (See Fig.2.1). We denote $\prod_{i=1}^{n}(u_i, G_i, .)$ by I_n^*.

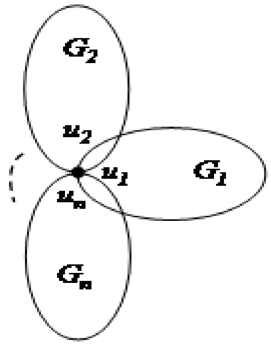

Fig. 2.1 The coalescence graph I_n^*

It is obvious that $p(I_n^*) = \sum_{i=1}^{n} p_i - n + 1$, $q(I_n^*) = \sum_{i=1}^{n} q_i$, and $diam(I_n^*) = max\{ \max_{1 \le i \le n}\{ diam G_i \}, \max_{i<j}\{ e_{G_i}(u_i) + e_{G_j}(u_j) \}\}$.

In the following theorem, we find the Hosoya polynomials of I_n^*.

Theorem 2.1: For any positive integer n, $n \ge 2$ we have

$$H(I_n^*; x) = \sum_{i=1}^{n} H(G_i; x) + \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} H(u_i, G_i; x) H(u_j, G_j; x) - (n-1)\sum_{i=1}^{n} H(u_i, G_i; x) + \frac{(n-1)(n-2)}{2}. \quad \ldots(2.1)$$

Proof: Using mathematical induction on n, $n \ge 2$.

For $n = 2$, we have by Gutman's Theorem (a),

$$H(I_2^*; x) = H(G_1; x) + H(G_2; x) + H(u_1, G_1; x) H(u_2, G_2; x) - H(u_1, G_1; x) - H(u_2, G_2; x).$$

Thus (2.1) is true for $n = 2$.

We assume that (2.1) is true for $n = r$, $r \ge 2$.

For $n = r+1$, again using Gutman's Theorem (a), we have

$$H(I_{r+1}^*; x) = H(I_r^*; x) + H(G_{r+1}; x) + H(u, I_r^*; x) H(u_{r+1}, G_{r+1}; x) - H(u, I_r^*; x) - H(u_{r+1}, G_{r+1}; x).$$

where $u = u_1 = u_2 = \ldots = u_r$.

From Fig. 2.1, we notice that

$$H(u, I_r^*; x) = \sum_{i=1}^{r} H(u_i, G_i; x) - (r-1)$$

Then $H(I_{r+1}^*; x) = H(I_r^*; x) + H(G_{r+1}; x)$

$$+ \left[\sum_{i=1}^{r} H(u_i, G_i; x) - (r-1) \right] H(u_{r+1}, G_{r+1}; x)$$

$$-\left[\sum_{i=1}^{r} H(u_i,G_i;x)-(r-1)\right]-H(u_{r+1},G_{r+1};x)\ .$$

By mathematical induction assumption , we have

$$H(I^*_{r+1};x)=\sum_{i=1}^{r} H(G_i;x)+\sum_{i=1}^{r-1}\sum_{j=i+1}^{r} H(u_i,G_i;x)H(u_j,G_j;x)$$

$$-r\sum_{i=1}^{r} H(u_i,G_i;x)+\frac{(r-1)(r-2)}{2}+H(G_{r+1};x)$$

$$+\sum_{i=1}^{r} H(u_i,G_i;x)H(u_{r+1},G_{r+1};x)-(r-1)H(u_{r+1},G_{r+1};x)$$

$$-\sum_{i=1}^{r+1} H(u_i,G_i;x)+(r-1)$$

$$=\sum_{i=1}^{r+1} H(G_i;x)+\sum_{i=1}^{r}\sum_{j=i+1}^{r+1} H(u_i,G_i;x)H(u_j,G_i;x)$$

$$-r\sum_{i=1}^{r+1} H(u_i,G_i;x)+\frac{r(r-1)}{2}\ .$$

Hence (2.1) is true for all n , $n \ge 2$. #

If each of G_1,G_2 , ... $,G_n$ is isomorphic to G , then $\prod_{i=1}^{n}(u,G,.)$ is denoted by $I^*_n(G)$, and $u_i = u$, for all $i = 1, 2, \ldots, n$ such that $u \in V(G)$.

Corollary 2.2: For $n \ge 2$, then

$$H(I^*_n(G);x)=nH(G;x)+\frac{n(n-1)}{2}\left[H(u,G;x)-1\right]^2-n+1. \qquad \#$$

3. Hosoya Polynomials of Bridges Coalescence Graphs:

Let w be isolated vertex , G_1,G_2 , ... $,G_n$ be disjoint connected graphs and $u_i \in V(G_i)$, $\forall i = 1,2,\ \ldots\ ,n$, then the **bridges coalenscence graph** $\prod_{i=1}^{n}(w,G_i,:)$ is obtained from $G_1,G_2,\ldots,G_n$ by adding new edge wu_i , for all $i = 1, 2, \ldots, n$.(See Fig.3.1). For simplicity we denote $\prod_{i=1}^{n}(w,G_i,:)$ by J^*_n .

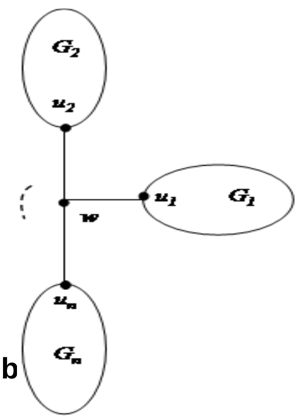

Fig. 3.1 The b graph J^*_n

It is clearly that $p(J_n^*)=\sum_{i=1}^{n}p(G_i)+1$, $q(J_n^*)=\sum_{i=1}^{n}q(G_i)+n,$ and $diam(J_n^*)=max\{\max_{1\le i\le n}\{diamG_i\},\max_{i<j}\{e_{G_i}(u_i)+e_{G_j}(u_j)\}+2\}$.

In the next theorem, we give the Hosoya polynomial of J_n^* for all $n\ge 2$.

Theorem 3.1: For any positive integer n , n≥ 2,we have:

$$H(J_n^*;x)=\sum_{i=1}^{n}H(G_i;x)+x\sum_{i=1}^{n}H(u_i,G_i;x)$$
$$+x^2\sum_{i=2}^{n}\sum_{j=1}^{i-1}H(u_j,G_j;x)H(u_i,G_i;x)+1 \quad \ldots(3.1)$$

Proof: From clearly that

$$H(J_1^*;x)=H(G_1;x)+xH(u_1,G_1;x)+1,$$

when $n=2$, then by Gutman's Theorem (b) , we get

$$H(J_2^*;x)=H(J_1^*;x)+H(G_2;x)+xH(w,J_1^*;x)H(u_2,G_2;x)$$
$$=H(G_1;x)+H(G_2;x)+xH(u_1,G_1;x)+1$$
$$+xH(w,J_1^*;x)H(u_2,G_2;x).$$

Since $H(w,J_1^*;x)=1+xH(u_1,G_1;x)$, then

$$H(J_2^*;x)=\sum_{i=1}^{2}H(G_i;x)+x\sum_{i=1}^{2}H(u_i,G_i;x)$$
$$+x^2H(u_1,G_1;x)H(u_2,G_2;x)+1.$$

This satisfied (3.1) when $n=2$.

Assume that (3.1) is true when $n=r,(r\ge 2)$.

Now , we shall prove that (3.1) is true , when $n=r+1$.

Since $H(J_{r+1}^*;x)=H(J_r^*;x)+H(G_{r+1};x)+xH(w,J_r^*;x)H(u_{r+1},G_{r+1};x)$, then

$$H(J_{r+1}^*;x)=\sum_{i=1}^{r}H(G_i;x)+x\sum_{i=1}^{r}H(u_i,G_i;x)$$
$$+x^2\sum_{i=2}^{r}\sum_{j=1}^{i-1}H(u_j,G_j;x)H(u_i,G_i;x)+1$$
$$+H(G_{r+1};x)+xH(w,j_r^*;x)H(u_{r+1},G_{r+1};x) . \quad \ldots(*)$$

From Fig. (3.1) , we notice that

$$H(w,J_r^*;x)=1+x\sum_{i=1}^{r}H(u_i,G_i;x). \quad \ldots(**)$$

Submitted (**) in (*) , we have

$$H(J_{r+1}^*;x)=\sum_{i=1}^{r+1}H(G_i;x)+x\sum_{i=1}^{r+1}H(u_i,G_i;x)$$
$$+x^2\sum_{i=2}^{r+1}\sum_{j=1}^{i-1}H(u_j,G_j;x)H(u_i,G_i;x)+1 .$$

This means that (3.1) hold when $n=r+1,\ r\ge 2$.

Hence (3.1) is true for all n , $n\ge 2$. #

If each of $G_1, G_2, \dots, G_n$ is isomorphic to G, and $H(u,G;x) = H(u_i,G_i;x)$, $u \in V(G)$ for all $i = 1,2,\dots n$, then $\prod_{i=1}^{n}(w,G,:)$ is denoted by $J_n^*(G)$.

Corollary 3.2: For $n \geq 2$, then

$$H(J_n^*(G);x) = nH(G;x) + nxH(u,G;x) + \frac{n(n-1)}{2}x^2(H(u,G;x))^2 + 1 . \ \#$$

4. Examples:

1. If K_t is the complete graph of order t, $t \geq 2$. then $diam K_t = 1$, $H(u,K_t;x) = 1 + (t-1)x$, $\forall\, u \in V(k_t)$, and $H(K_t;x) = t + \frac{1}{2}t(t-1)x$.

Thus , from Corollaries 2.2, and 3.2 , we have

$$H(I_n^*(K_t);x) = (nt - n + 1) + \frac{n}{2}t(t-1)x + \frac{n(n-1)}{2}(t-1)^2x^2 .$$

$$H(J_n^*(K_t);x) = (nt+1) + (\frac{n}{2}t(t-1) + n)x + (n(t-1) + \frac{n(n-1)}{2})x^2$$

$$+ \left(n(n-1)(t-1)\right)x^3 + (\frac{n(n-1)}{2}(t-1)^2)x^4 .$$

2. If C_t is an even cycle graph of order t, $t = 2m, m \geq 2$, then , $diam\, C_{2m} = m$, $H(u,C_{2m};x) = 1 + 2\sum_{r=1}^{m-1}x^r + x^m$, $\forall\, u \in V(C_{2m})$, and $H(C_{2m};x) = 2m\sum_{r=0}^{m-1}x^r + mx^m$.

Thus ,from Corollaries 2.2, and 3.2 , we have

$$H(I_n^*(C_{2m});x) = 2mn\sum_{r=0}^{m-1}x^r + mnx^m + \frac{n(n-1)}{2}F(x) - n + 1 ,$$

where $F(x) = 4\sum_{r=1}^{m-1}rx^{r+1} + 4\sum_{r=1}^{m-1}(m-r)x^{m+r} + x^{2m}$.

$$H(J_n^*(C_{2m});x) = 2mn\sum_{r=0}^{m-1}x^r + mnx^m + nx(1 + 2\sum_{r=1}^{m-1}x^r + x^m) + \frac{n(n-1)}{2}x^2F(x) + 1,$$

where $F(x) = 4\sum_{r=1}^{m-1}(rx+1)x^r + 4\sum_{r=1}^{m-1}(m-r)x^{m+r} + (1+x^m)^2$.

3. If C_t is an odd cycle graph of order $t, t = 2m-1, m \geq 2$,then $diam\, C_{2m-1} = m-1$, $H(u,C_{2m-1};x) = 1 + 2\sum_{r=1}^{m-1}x^r$, $\forall u \in V(C_{2m-1})$, and $H(C_{2m-1};x) = (2t-1)\sum_{r=0}^{m-1}x^r$.

Then using Corollaries 2.2, and 3.2 , we have

$$H(I_n^*(C_{2m-1});x) = 2nm\sum_{r=0}^{m-1}x^r - n\sum_{r=0}^{m-1}x^r + n(n-1)F(x) - n + 1 ,$$

where $F(x) = 4\sum_{r=1}^{m-1}rx^{r+1} + 4\sum_{r=1}^{m-2}(m-r-1)x^{r+m}$.

$$H(J_n^*(C_{2m-1});x) = 2nm\sum_{r=0}^{m-1}x^r - n\sum_{r=0}^{m-1}x^r + nx + 2n\sum_{r=1}^{m-1}x^{r+1} + \frac{n(n-1)}{2}x^2F(x) + 1,$$

where $F(x) = 1 + 4\sum_{r=1}^{m-1}x^r + 4\sum_{r=1}^{m-1}rx^{r+1} + 4\sum_{r=1}^{t-2}(t-r-1)x^{r+m}$.

4. Let S_t be the star of order t, $t \ge 4$, with the vertex set $V(S_t)=\{c, v_1, v_2, \dots, v_{t-1}\}$, where $deg\, c = t-1$ and $deg\, v = 1$, $\forall\, v \in V(S_t)-\{c\}$. Since $diam S_t = 2$, $H(v,S_t;x)=1+x+(t-2)x^2$, for all $v \in V(S_t)-\{c\}$, and $H(S_t;x)=t+(t-1)x+\frac{1}{2}(t-1)(t-2)x^2$,

then using Corollaries 2.2, and 3.2 , where the coalescence vertex is not c, we have

$$H(I_n^*(S_t);x)=(nt-n+1)+n(t-1)x+\frac{n}{2}((t-1)(t-2)+n-1)x^2$$
$$+n(n-1)(t-2)x^3+\frac{n(n-1)}{2}(t-2)^2x^4 .$$
$$H(J_n^*(S_t);x)=(nt+1)+nt\,x+\frac{n}{2}((t-1)(t-2)+n+1)x^2+n(t+n-3)x^3$$
$$+\frac{n}{2}(n-1)(2t-3)x^4+n(n-1)(t-2)x^5+\frac{n}{2}(n-1)(t-2)^2x^6 .$$

5. Let W_t be the wheel graph of order t, $t \ge 4$, with vertex set $V(W_t)=\{c, v_1, v_2, \dots, v_{t-1}\}$, where $deg\, c = t-1$, and $deg\, v = 3$, $\forall\, v \in V(W_t)-\{c\}$. Since $diam W_t = 2$, $H(v,W_t;x)=1+3x+(t-4)x^2$, for all $v \in V(W_t)-\{c\}$ and $H(W_t;x)=t+2(t-1)x+\frac{(t-1)(t-4)}{2}x^2$,

then using Corollary 2.2 , where the coalescence vertex is not c ,we get:

$$H(I_n^*(W_t);x)=(nt-n+1)+2n(t-1)x+\frac{n}{2}\left((t-1)(t-4)+9(n-1)\right)x^2$$
$$+3n(n-1)(t-4)x^3+\frac{n}{2}(n-1)(t-4)^2x^4 .$$
$$H(J_n^*(W_t);x)=(nt+1)+n(2t-1)x+\frac{n}{2}\left((t-1)(t-4)+n+5\right)x^2+n(t+3n-7)x^3$$
$$+\frac{n(n-1)}{2}(2t+1)x^4+3n(n-1)(t-4)x^5+\frac{n}{2}(n-1)(t-4)^2x^6 .$$

5. Wiener index and average distance.

The Wiener index of a connected graph G is denoted by $W(G)$ and defined by : $W(G)=\frac{1}{2}\sum_{u,v\in V(G)} d(u,v).$ ...(5.1)

The name Wiener index for the quantity defined in (5.1) is usual in chemical literature, since Harold Wiener [9] in 1947 seems to be the first who considered it. Several mathematical authors obtained the Wiener index of many kinds of chain of cycle graphs [2,7]. The Wiener index of G can be obtained from the following formula [5]

$$W(G)=\frac{d}{dx}H(G;x)\Big|_{x=1}=\sum_{k\ge 1} k\,d(G,k), \qquad ...(5.2)$$

But The Wiener index of a vertex u in G can be obtained from the following formula [1]

$$W(u,G)=\frac{d}{dx}H(u,G;x)\Big|_{x=1}=\sum_{k\ge 1} k\,d(u,G,k), \qquad ...(5.3)$$

Now from (5.2) and (5.3), we obtain the following corollary:

Corollary 5.1: Let G be a connected graph with order p, then for $n \geq 2$, the Wiener index of $I_n^*(G)$ and $J_n^*(G)$ are given by :

$$W(I_n^*(G)) = nW(G) + n(n-1)(p-1)W(u,G).$$

$$W(J_n^*(G)) = nW(G) + n((n-1)p+1)(p+W(u,G)). \qquad \#$$

Now from Corollary (5.1) and Examples 1-5 in Section 4 , we obtain the following corollary:

Corollary 5.2: For $n \geq 3$,we have:

1. $W(I_n^*(K_t)) = \frac{n}{2}[(2n-1)t^2 - (4n-3)t + 2(n-1)]$,

   $W(J_n^*(K_t)) = \frac{n}{2}[(4n-3)t^2 - (2n-5)t - 2]$.
2. $W(I_n^*(C_{2m})) = n(2n-1)m^3 - n(n-1)m^2$,

   $W(J_n^*(C_{2m})) = nm[(2n-1)m^2 + (4n-3)m + 2]$.
3. $W(I_n^*(C_{2m-1})) = \frac{mn}{2}\left[2(2n-1)m^2 - (8n-5)m + 4n - 3\right]$,

   $W(J_n^*(C_{2m-1})) = \frac{n}{2}[(4n-2)m^3 + (2n-3)m^2 - (6n-9)m + (2n-4)]$.
4. $W(I_n^*(S_t)) = n(2n-1)t^2 - n(5n-3)t + n(3n-2)$,

   $W(J_n^*(S_t)) = n[(3n-2)t^2 - (3n-4)t - 2]$.
5. $W(J_n^*(W_t)) = n[(3n-2)t^2 - 5(n-1)t - 3]$,

   $W(I_n^*(W_t)) = n(2n-1)t^2 - n(7n-4)t + n(5n-3)$. #

The main distance or average distance of a connected graph G, denoted by $\mu(G)$ and is defined by

$$\mu(G) = 2W(G)/p(p-1), \quad p = |V(G)|. \qquad \ldots(5.4)$$

Finally, from Corollary 5.2 and (5.4) , we have the following corollary :

Corollary 5.3: For $n \geq 3$,we have:

1. $\mu(I_n^*(K_t)) = 2 - \frac{t}{nt-n+1}$,

   $\mu(J_n^*(K_t)) = 4 - (\frac{2}{t} + \frac{3t-1}{nt+1})$.
2. $\mu(I_n^*(C_{2m})) = m - \frac{m(2m^2 - 2nm + n - 1)}{4nm^2 - 4nm + 2m + n - 1}$,

   $\mu(J_n^*(C_{2m})) = m + 2 - \frac{m(m+n)}{2nm+1}$.
3. $\mu(I_n^*(C_{2m-1})) = m - \frac{m(2m^2 - 3m + 1)}{2(2nm^2 - 4nm + m + 2n - 1)}$,

   $\mu(J_n^*(C_{2m-1})) = m + \frac{3}{2} - \frac{4m^3 + 10m^2 + 2(n-7)m - n + 5}{2(2m-1)(2nm-n+1)}$.
4. $\mu(J_n^*(S_t)) = 6 - \frac{6}{t} - \frac{2(2t+1)(t-1)}{t(nt+1)}$,

$$\mu(I_n^*(S_t)) = 4 - \frac{2(t+n)}{nt-n+1}.$$

5. $$\mu(I_n^*(W_t)) = 4 - \frac{2(t+3n-1)}{nt-n+1},$$

$$\mu(J_n^*(W_t)) = 6 - \frac{10}{t} - \frac{4(t^2-t-1)}{t(nt+1)}. \quad \#$$

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