NEW UPPER AND LOWER BOUNDS FOR LAPLACIAN EIGENVALUES OF GRAPHS

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Abstract

In this article, we present lower bounds for the Laplacian spectral radius, λ_1 and the second largest eigenvalue, λ_2 of the Laplacian matrix , L , of a simple connected graph in terms of basic invariants of the graph. Also, we use a well-known result to obtain an upper bound for λ_1 in terms of traces of powers of L. Finally, we apply these bounds to some examples of graphs and make a comparison with some known bounds.

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1 INTRODUCTION

Let *G*=(*V, E*) be a simple connected graph with vertex set $V=\{v_1,v_2,\ldots,v_n\}$ and edge set *E*. For $v_i \in V$, let the degree of v_i and the average of the degrees of the vertices adjacent to v_i be denoted by d_i and m_i, respectively and let n_{ij} denotes the number of common neighbors of v_i and v_j. We write *i~j* to indicate that the edge $v_iv_j \in E$. Let *A* be the adjacency matrix of *G* and *D* be the diagonal matrix of vertex degrees. The Laplacian matrix of *G* is $L(G)=L=D-A$. Clearly, *L* is symmetric and positive semi-definite and consequently its eigenvalues, which are called the Laplacian eigenvalues of *G*, are real nonnegative numbers and since each row sum of *L* is equal to 0, then 0 is the smallest eigenvalue of *L.* Denote the eigenvalues of *L* by $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \geq 0$.

The spectrum of the Laplacian matrix is important in graph theory because it has a close relation to numerous graph invariants, such as diameter, chromatic number, maximum cut, spanning trees, connectivity, etc. (for more details, see [1, 2] and the references therein).Thus, good lower and upper bounds for the eigenvalues of *L* are needed in many applications.

Among the known lower bounds for the Laplacian spectral radius, λ_1, are the following:

(1) R. Grone and R. Merris' bound [3]

$$\lambda_1 \geq 1+\max_{v_i\in V} d_i \tag{1}$$

(2) K. C. Das' bound [4]

$$\lambda_1 \geq \frac{1}{\sqrt{2}}\max_{i\sim j}\sqrt{(d_i^2+2d_i-2d_j-2)+\sqrt{(d_i^2+2d_i+2d_j+4)^2+4(d_i-n_{ij}-1)(d_j-n_{ij}-1)}} \tag{2}$$

(3) For triangle-free graphs, X.D. Zhang and R. Luo's bound [5]

$$\lambda_1 \geq \tfrac{1}{2}\max_{v_i\in V}\left\{(d_i+m_i)+\sqrt{(d_i-m_i)^2+4d_i}\right\} \tag{3}$$

Among the known upper bounds for the spectral radius, λ_1, are the following:

(1) R. Merris' bound [6]

$$\lambda_1 \leq \max_{v_i\in V}\{d_i+m_i\} \tag{4}$$

(2) K. C. Das' bound [7]

$$\lambda_1 \leq \tfrac{1}{2}\max_{i\sim j}\left\{d_i+d_j+\sqrt{(d_i-d_j)^2+4m_im_j}\right\} \tag{5}$$

(3) T. Wang, J. Yang and B. Li's bound [8]

$$\lambda_1 \leq \max_{v_i\in V}\left\{d_i+\frac{d_i\left(m_i+\sqrt{m_i}\right)}{d_i+\sqrt{d_i}}\right\} \tag{6}$$

In this paper, we derive lower bounds for the Laplacian spectral radius, λ_1 , and the second largest eigenvalue, λ_2 ,in terms of basic invariants of the graph. We also imploy one well-known result to compute an upper bound for λ_1 in terms of traces of powers of *L*.Finally, we apply the above mentioned bounds and our bounds on some examples of graphs to conclude that our bounds are good in some sense.

2 MAIN RESULTS

First, we obtain lower bounds for the Laplacian eigenvalues λ_1 and λ_2 ,of a simple graph by employing the following result, known as the Cauchy interlacing property. For a proof, see [9, p. 189].

Lemma 1. If *B* is an $n \times n$ symmetric matrix and $\tilde{B}$ is an $m \times m$ principal submatrix of *B* with eigenvalues $\lambda_1(B) \geq \ldots \geq \lambda_n(B)$ and $\lambda_1(\tilde{B}) \geq \ldots \geq \lambda_m(\tilde{B})$, respectively, then the eigenvalues of *B* interlace those of $\tilde{B}$, that is

$$\lambda_i(B) \geq \lambda_i(\tilde{B}) \geq \lambda_{n-m+i}(B) \quad \text{for } i = 1, \ldots, m.$$

Theorem 1. Let *G* be a simple connected graph and let λ_1 and λ_2 be the largest and the second
largest eigenvalues of $L(G) = (l_{ij})$. Then

$$\lambda_1 \geq \frac{1}{\sqrt{2}} \max_{i<j} \sqrt{\delta_i + \delta_j + \sqrt{(\delta_i - \delta_j)^2 + 4m_{ij}^2}} \tag{7}$$

and

$$\lambda_2 \geq \frac{1}{\sqrt{2}} \max_{i<j} \sqrt{\delta_i + \delta_j - \sqrt{(\delta_i - \delta_j)^2 + 4m_{ij}^2}} \,, \tag{8}$$

where

$$\delta_i = d_i^{\,2} + d_i \quad \text{and} \quad m_{ij} = (d_i + d_j)\, l_{ij} + n_{ij} \,.$$

Moreover, the equality in (7) holds if *G* is a complete graph K_n .

Proof . Let $\tilde{L}_{ij}$ be the 2×2 principal submatrix of L^2 corresponding to two vetrtices $v_i, v_j \in V$, then $\tilde{L}_{ij}$ has the form

$$\tilde{L}_{ij} = \begin{bmatrix} l_i^{\,t} l_i & l_i^{\,t} l_j \\ l_j^{\,t} l_i & l_j^{\,t} l_j \end{bmatrix} ,$$

where l_i is the i^{th} column vector of L and the superscript t denotes transposition.
It is easy to show that

$$\widetilde{L}_{ij} = \begin{bmatrix} \delta_i & m_{ij} \\ m_{ij} & \delta_j \end{bmatrix}.$$

Since $\lambda_i(L) = \sqrt{\lambda_i(L^2)}$ and, by Lemma 1, the eigenvalues of L^2 interlace those of $\widetilde{L}_{ij}$ then by direct computation of the two eigenvalues of $\widetilde{L}_{ij}$, the inequalities in (7) and (8) follow.
Now, if $G = K_n$, then $\delta_i = \delta_j = n(n-1)$ and $m_{ij} = -n$, so the lower bound in (7) is equal to $n = \lambda_1(K_n)$. □

Let ℓ be our lower bound in (7). The following lemma present a case where ℓ is better than a well-known lower bound for λ_1.

Lemma 2. Let *G* be a simple connected graph and let v_i and v_j be the two vertices with largest degrees. If v_i and v_j are adjacent and have equal degrees, say $d_i = d_j = d$, then

$$\ell \geq 1 + d \quad . \tag{9}$$

Proof . Clearly, we will have

$$\ell \geq \frac{1}{\sqrt{2}} \sqrt{2d(d+1) + \sqrt{4m^2_{ij}}} \quad .$$

Since $m_{ij} = n_{ij} - 2d$ and $d - n_{ij} \geq 1$, we obtain that (9) holds. □

Remark 1. Note that the right hand side in (9) is the lower bound in (1) due to Grone and Merris [3].

Next we obtain an upper bound for the Laplacian spectral radius, λ_1, of a graph by employing the following well-known result. For a proof, see [11].

Lemma 3. Let B be an $n \times n$ symmetric matrix with eigenvalues $\lambda_1 \geq \ldots \geq \lambda_n$ and let

$$m = \frac{tr(B)}{n} \qquad \text{and} \qquad s^2 = \frac{tr(B^2)}{n} - m^2 \quad ,$$

where $tr(B)$ stands for the trace of B. Then

$$\lambda_1 \leq m + s\sqrt{n-1} \quad . \tag{10}$$

Theorem 2. Let *G* be a simple graph of order *n*. Then

$$\lambda_1 \leq \sqrt{m + s\sqrt{n-1}}\,, \tag{11}$$

where

$$m = \frac{1}{n}\sum_{i=1}^{n} \delta_i \qquad \text{and} \qquad s^2 = \frac{1}{n}\left[\sum_{i=1}^{n} \delta^2{}_i + 2\sum_{i<j} m^2{}_{ij}\right] - m^2 \quad .$$

Proof . Clearly,

$$(L^2)_{ij} = \left\{ \begin{array}{ll} \delta_i & , \ i = j \\ m_{ij} & , \ i \neq j \end{array} \right\}.$$

So, one can easily see that

$$tr(L^2) = \sum_{i=1}^{n} \delta_i \qquad \text{and} \qquad tr(L^4) = \sum_{i=1}^{n} \delta^2{}_i + 2\sum_{i<j} m^2{}_{ij} \ .$$

Since L^2 is symmetric and $\lambda_1(L) = \sqrt{\lambda_1(L^2)}$, the inequality in (11) follows by applying Lemma 2 to $B = L^2$. □

In the following lemma, we investigate the case where the equality in (10) holds and show that this imposes a certain relation between the eigenvalues of the matrix *B*.

Lemma 4. The equality in (10) holds if and only if $\lambda_2 = \ldots = \lambda_n$.

Proof . From the definition of s^2 in Theorem 1, from the fact that the sum of the eigenvalues of a matrix is equal to its trace and by using Lagrange's identity, we have

$$s^2 = \frac{1}{n^2}\left[n\sum_{i=1}^{n} \lambda_i^{\,2} - \left(\sum_{i=1}^{n} \lambda_i\right)^2 \right] = \frac{1}{n^2}\sum_{1\le i<j\le n}\left(\lambda_i - \lambda_j\right)^2 .$$

Therefore, the equality in (10) is equivalent to

$$\lambda_1 = \frac{1}{n}\left[\sum_{i=1}^{n} \lambda_i + \sqrt{(n-1)\sum_{1\le i<j\le n}\left(\lambda_i - \lambda_j\right)^2}\,\right] \quad ,$$

which is equivalent to

$$\left[(n-1)\lambda_1 - \sum_{i=2}^{n} \lambda_i\right]^2 = (n-1)\sum_{1\le i<j\le n}\left(\lambda_i - \lambda_j\right)^2 \quad ,$$

which is the same as

$$\left[\sum_{j=2}^{n} \left(\lambda_1 - \lambda_j\right)\right]^2 = (n-1)\sum_{1\le i<j\le n}\left(\lambda_i - \lambda_j\right)^2 \quad . \tag{12}$$

Since, by Cauchy-Schwartz inequality, we have

$$\left[\sum_{j=2}^{n}(\lambda_1-\lambda_j)\right]^2 \le (n-1)\sum_{j=2}^{n}(\lambda_1-\lambda_j)^2 ,$$

then it is easy to see that the equality (12) holds if and only if $\lambda_2 = \ldots = \lambda_n$. □

3 EXAMPLES

As the conclusion of this article, we give some examples to illustrate our results. Consider the three graphs shown in the figure below.

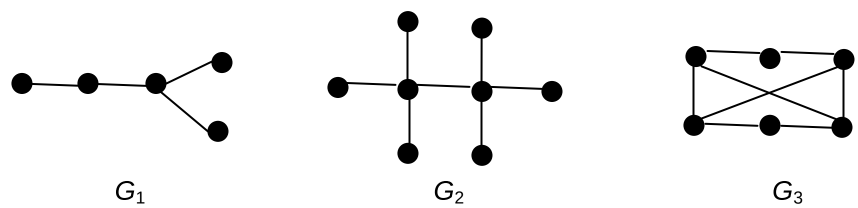

Fig. 1

The actual values of λ_1 and λ_2 and their bounds are as shown in the tables below.

Table 1: The values of λ_1 and the lower bounds in (1)-(3) and (7)

	λ_1	(1)	(2)	(3)	(7)
G_1	4.17	4	4.01	4.08	3.85
G_2	5.65	5	4.02	5.16	5.29
G_3	5.56	4	4.01	4.57	4.24

Table 2: The values of λ_2 and the lower bound in (8)

	λ_2	(8)
G_1	2.31	1.78
G_2	4	3.46
G_3	3	2.45

Table 3: The values of λ_1 and the upper bounds in (4)-(6) and (10)

	λ_1	(1)	(2)	(3)	(10)
G_1	4.17	4.33	4.21	4.57	4.24
G_2	5.65	5.75	5.75	6.05	5.75
G_3	5.56	5.66	5.67	5.93	5.67

Remark 2. From the tables above, we can see that in some cases our bounds are better than some well-known bounds and so, they are good in some sense.

Remark 3. Note that Lemma 2 applies to G_2 and G_3 and also to any regular graph. For example, for the Petersen graph, shown in the figure below , which is a 3-regular graph and for which $\lambda_1 = 5$, applying (7), we get $\lambda_1 \geq 4.24$. However, applying (1) and (2), we have $\lambda_1 \geq 4$ and $\lambda_1 \geq 4.02$ **,** respectively**.** This adds an example where our bound in (7) is better than some known lower bounds for λ_1 .

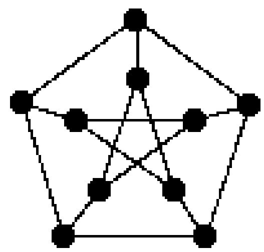

Fig. 2: The Petersen graph

Remark 4. For the interested reader, in [12] we give good estimates of a lower bound for the Laplacian spectral radius, λ_1, of triangle-free graphs that is better than most of the known lower bounds for λ_1**.**

Remark 5. We would like to note that the equality in (10) does not hold for the Laplacian matrix of a connected graph since it is well-known that the algebraic connectivity of a connected graph which is the second smallest Laplacian eigenvalue is greater than zero, that is, $\lambda_{n-1} > 0 = \lambda_n$.

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