Shortest Path in Kleene Algebra

M. Al-Mosa, F. Tchier and F. Toufic
Mathematics department,
King Saud University
malmousa@ksu.edu.sa, ftchier@ksu.edu.sa,
ftoufic@ksu.edu.sa

***Abstract*—Using Kleene algebra and its properties. We will find the shortest path which links two given vertices in a directed (finite) graph. Also, we will give an algorithm to solve this problem by using the typed Kleene algebra.**

***Index Terms*—Kleene algebra, shortest path, graph, relation.**

I. Introduction

The shortest path problem consists of finding a path between two vertices such that the sum of the weights of its constituent edges is minimized. Many important algorithms have been used to solve this problem; Dijkstra's algorithm solves the single-pair, single-source, and single-destination shortest path problems [2]. Bellman-Ford algorithm solves the single source problem if edge weights may be negative [12]. Floyd-Warshall algorithm solves all pairs shortest paths [2], [12]. Johnson's algorithm solves all pairs shortest paths, and may be faster than Floyd-Warshall on sparse graphs [2], [12], [17]. Perturbation theory finds (at worst) the locally shortest path. In this paper, we will use Kleene algebra to solve this problem.

In the following we will present some notions needed for the rest of the paper.

(1) **Definition.**

1. A partial order is a binary relation "$\leq$" over a set S which is reflexive, antisymmetric, and transitive, that is, for all a, b, and $c \in S$, we have
   - $a \leq a$ (reflexivity);
   - if $a \leq b$ and $b \leq a$ then $a = b$ (antisymmetric);
   - if $a \leq b$ and $b \leq c$ then $a \leq c$ (transitivity).
2. A set with a partial order is called a partially ordered set (also called a poset).
3. Let $(A, \leq)$ and $(B, \leq)$ be two partially ordered sets. A Galois connection between these posets consists of two functions: $F : A \to B$ and $G : B \to A$, such that for all a in A and b in B, we have $F(a) \leq b$ if and only if $a \leq G(b)$.

Now we will give the definition of directed graph [2].

(2) **Definition.** A directed graph $G = (V, B)$ consists of a set V of vertices and a set B of ordered pairs of vertices, called edges. And the adjacency matrix of G is a means of representing which vertices of a graph are adjacent to which other vertices.

II. Kleene algebra

Kleene algebra is an algebraic structure that captures axiomatically the properties of a natural class of structures arising in logic and computer science. It is named after Stephen Cole Kleene $(1909 - 1994)$, who among his many other achievements invented finite automate and regular expressions, structures of fundamental importance in computer science. Kleene algebra is the algebraic theory of these objects, although it has many other natural and useful interpretations.

Kleene algebras arise in various guises in many contexts; relational algebra [1], [3], [4], semantics, logic of programs [5], [6], automata, formal language theory [7], [8], the design and analysis of algorithms [9], [10], [11], [12].

(3) **Definition.** A Kleene algebra is a structure $\mathcal{K} = (K, +, \cdot, *, 0, 1)$ where the following axioms satisfied

- $(a + b) + c = a + (b + c)$
- $a + a = a$
- $a + b = b + a$
- $a + 0 = a$
- $a.0 = 0.a = 0$
- $(ab)c = a(bc)$
- $a1 = 1a = a$
- $a(b + c) = ab + ac$
- $(a + b) \cdot c = ac + bc$.

$$
\begin{array}{ll}
(K1) & 1 + xx^* \leq x^* \\
(K2) & 1 + x^*x \leq x^* \\
(K3) & b + ax \leq x \text{ then } \ a^*b \leq x \\
(K4) & b + xa \leq x \text{ then } \ ba^* \leq x.
\end{array}
$$

where $a \leq b$ iff $a + b = b$ is called the natural partial order of Kleene algebra.

(4) **Definition.** A Kleene algebra $\mathcal{K}$ is called star-continuous [16] if it satisfies the axiom

$xy^*z = \sup_{n \geq 0} xy^nz$,

for any $x, y, z \in \mathcal{K}$; where $y^0 = 1$, $y^{n+1} = yy^n$ and the supremum is with respect to the natural order.

(5) **Example.** (Typed Kleene algebra). Let Σ^* denote the set of finite-length strings over a finite alphabet Σ, including the null string ε. The set Σ^* forms a Kleene algebra under the following constants and operations on subsets of Σ^*

1. $A+B=A\cup B$,
2. $A.B=A\bowtie B=\{x\bullet z\bullet y \mid x\bullet z\in A,\ z\bullet y\in B\}$, (where $\bullet$ is concatenation)
3. $0=\emptyset$,
4. $1=\{\varepsilon\}$,
5. $A^*=\bigcup_{n\geq 0}A^n = \{x_1...x_n \mid n\geq 0 \text{ and } x_i\in A, 1\leq i\leq n\}$.

Thus the operation $(\cdot)$, applied to two sets of strings A and B, products the set of all strings obtained by concatenating a string from A with a string from B, in the order. Thus A^* is the union of all powers of A, equivalently, A^* consists of all strings obtained by concatenating together any finite collection of string from A in any order. Any subset of the full power set of Σ^* containing $\emptyset$ and $\{\varepsilon\}$ and closed under the operations of $(\cup),(\cdot)$, and $(^*)$ is a Kleene algebra.

A subidentity of a Kleene algebra is an element x with $x\leq 1$. We call a Kleene algebra pre-typed if all its subidentities are idempotent, *i.e.*, $x\leq 1 \Longrightarrow x\bullet x=x$. We call a pretyped Kleene algebra typed if its a boolean algebra and the restriction operations distribute through arbitrary meets of subtypes, *i.e.*, if we have for all families $(x_j)_{j\in J}$ of subidentites and all $a\in S$ that

$$(\bigcap_{j\in J}x_j)\cdot a=\bigcap_{j\in J}x_j\cdot a \text{ and } a\cdot(\bigcap_{j\in J}x_j)=\bigcap_{j\in J}a\cdot x_j.$$

In a typed Kleene algebra we can define, for $a\in S$, the domain $\langle a$ and co-domain $a\rangle$ via the Galois connections (y ranges over subidentities only)

$$\begin{aligned}\langle a\leq y &\Longleftrightarrow a\leq y\cdot 1,\\ a\rangle\leq y &\Longleftrightarrow a\leq 1\cdot y.\end{aligned}$$

Now, we take unusual model that turns out to be useful in the shortest path algorithms in graphs. This algebra is called the tropical algebra, also known as the "min, + algebra". For more details see [17].

(6) **Example.** (The tropical algebra). Let $\mathcal{R}=R^+\cup\{\infty\}$ is the set of nonnegative reals with an additional infinite element ∞. This model forms a Kleene algebra under the following constants and operations on subsets of $R^+\cup\{\infty\}$:

1. $a+b=\min\{a,b\}$,
2. $a\cdot b=a+_R b$, (where $+_R$ means the addition in reals)
3. $a^*=1=0_R$,
4. $0=\infty$,
5. $1=0_R$.

The elements 1 and 0 of a Kleene algebra can play the roles of the truth values "true" and "false". Expressions that yield one of these values are therefore also called assertions. The assertion 0 means not only "false", but also "undefined". Negation is defined by $\sim 0=1$, and $\sim 1=0$. Then for an assertion b and an element c we have

$$b\cdot c=c\cdot b=\begin{cases} c & \text{if } b=1,\\ 0 & \text{if } b=0.\end{cases}$$

The conjunction of assertions a,b is their infimum $a\cap b$ or, equivalently, their product $a\cdot b$; their disjunction is their sum $a+b$. We write $a\wedge b$ for $a\cap b$ and $a\vee b$ for $a+b$.

Using this, we can construct a conditional:

$$\text{if } b \text{ then } c \text{ else } d \text{ fi}=b\cdot c\cup\ \sim b\cdot d$$

for assertions b and elements c,d. Note that the conditional is monotonic only in d and e. So, recursions over the condition b need not be well-defined. A property we are going to use in the sequel is

$$\text{if } b \text{ then } d \text{ else if } c \text{ then } d \text{ else } e \text{ fi fi} = \text{if } b\vee c \text{ then } d \text{ else } e \text{ fi} \quad (1)$$

For assertions b,c and elements d,e.

A. Matrices Over a Kleene Algebra

Under the natural definitions of the Kleene algebra operators $+,\cdot,^*,0$ and 1, the family $\mathcal{M}(n,\mathcal{K})$ of $n\times n$ matrices over a Kleene algebra $\mathcal{K}$ again forms a Kleene algebra. This is a standard result proved for various classes of algebras in [13], [14].

Define $(+)$ and $(\cdot)$ on $\mathcal{M}(n,\mathcal{K})$ to be the usual operations of matrix addition and multiplication, respectively, Z_n the $n\times n$ zero matrix, and I_n the $n\times n$ identity matrix. The partial order $(\leq)$ is defined on $\mathcal{M}(n,\mathcal{K})$ by $A\leq B\Longleftrightarrow A+B=B$.

The structure $(\mathcal{M}(n,\mathcal{K}),+,\cdot,Z_n,I_n)$ is an idempotent semiring.

The definition of E^* for $E\in\mathcal{M}(n,\mathcal{K})$ comes from [8], [13], [15]. We first consider the case $n=2$. This construction will later be applied inductively.

Let

$$E=\begin{pmatrix} a & b\\ c & d\end{pmatrix}$$

let $f=a+bd^*c$ and define

$$E^*=\begin{pmatrix} f^* & f^*bd^*\\ d^*cf^* & d^*+d^*cf^*bd^*\end{pmatrix}$$

The matrix E^* defined above satisfies the Kleene algebra axioms (Def. 3).

If $\mathcal{K}$ is star-continuous (Def. 4), then so is $\mathcal{M}(n,\mathcal{K})$ for $n\geq 1$.

This Proposition states that the star-continuity `passes` from the Kleene algebra $\mathcal{K}$ to $\mathcal{M}(n,\mathcal{K})$ and this is exactly the key fact that we use in the following application.

B. All Pair-Shortest Paths Problem

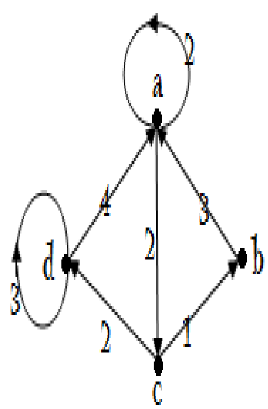
[11]

Given a directed graph G which has n vertices $1,...,n$ and each edge is labelled by a positive real number called the weight of the edge. From G, we create a complete graph [12] (still denoted by G) such that the vertex i is connected with j (in this order) by an edge labelled with the same number if i and j are adjacent in G (in this order again), and by an edge labelled by the symbol ∞ otherwise. So, the edges of the new graph G are labelled by an element of $R^+ \cup \{\infty\}$. In the set $\mathcal{R} = R^+ \cup \{\infty\}$, we define the operations as in Example 6.

The adjacency matrix B of the previous graph G is an element of the Kleene algebra $\mathcal{M}(n,\mathcal{K})$ and the ij^{th} entry of B^k is exactly the number of paths of length k which links i with j and composed by less than or equal to k edges.This result can be used to determine the length of the shortest path between i and j. By the proposition A,
$B^* = \sup_{\mathcal{R}} \{B^k \mid k \in N\} = \inf\{B^k \mid k \in N\}$.

So the ij^{th} entry of B^* is exactly the length of the shortest path which links i to j.

(7) **Example.** The adjacency matrix of the following graph is



$$B = \begin{pmatrix} 2 & \infty & 2 & \infty \\ 3 & \infty & \infty & \infty \\ \infty & 1 & \infty & 2 \\ 4 & \infty & \infty & 3 \end{pmatrix} = \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix}.$$

We have $B_4^* = \begin{pmatrix} 0 & 2 \\ \infty & 0 \end{pmatrix}$

,F=$\mathrm{B}_1 + B_2 B_4^* B_3 = \begin{pmatrix} 2 & 3 \\ 3 & \infty \end{pmatrix}$ and $F^* = \begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix}$.
Therefore, with a bit of calculation we find

$$B^* = \begin{pmatrix} F^* & F^* B_2 B_4^* \\ B_4^* B_3 F^* & B_4^* + B_4^* B_3 F^* B_2 B_4^* \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 3 & 2 & 4 \\ 3 & 0 & 5 & 7 \\ 4 & 1 & 0 & 2 \\ 4 & 7 & 6 & 0 \end{pmatrix}.$$

So, we can read on this matrix that the shortest path from the vertex b to the vertex d is of length $(B^*)_{bd} = 7$, it passes through a and after c.

In the following, we give an algorithm to solve the shortest path problem by using the typed Kleene algebra.

C. Shortest Connecting Path

Assume that $(S, \Sigma, \cdot, 0, 1)$ is a typed Kleene algebra. We define the general operation E by $E(W)(f,g) = g(f(w))$ where

- $w \subseteq S$ is a fixed element of S.
- $f : S \longrightarrow \mathcal{P}(M)$ is a disjunctive abstraction function with some set M of "valuations", where a function f from a Kleene algebra into a lattice is disjunctive, if it distributes through $+$, *i.e.*, satisfies $f(x+y) = f(x) \cup f(y)$,
- $g : \mathcal{P}(M) \longrightarrow \mathcal{P}(M)$ is a selection satisfying the properties

$$(1) \qquad g(K) \subseteq K,$$
$$(2) \ g(K \cup L) = g(g(K) \cup g(L))$$

(weak distributivity),
for $K, L \subseteq M$.

Motivated by the graph theoretical applications, we now postulate the following conditions about f and g:

1. $\langle c \leq a \rangle \Rightarrow$
$g(f(a \cdot c)) = g(f(a \cdot c) + f(a \cdot u \cdot c))$,
2. $(a \cdot u)\rangle \leq a\rangle \Rightarrow$
$g(f(a \cdot c)) = g(f(a \cdot c) + f(a \cdot u \cdot c))$,

with $a, c, u \in S$. These two conditions are used to obtain two termination cases of the algorithm. For more details see [17].

We have the following basic algorithm:

$$\begin{array}{c} F(f,g)(a,b,c) = if\langle c \leq a\rangle \ \vee \ (a \cdot b)\rangle \leq a\rangle \\ then\ g(f(a \cdot c)) \\ else\ g(f(a \cdot c)) + F(f,g)(a + a \cdot b, b, c))fi \end{array} \qquad (2)$$

We define

$$shortestpaths(a,c) {=} F(id, \min paths)(a,c), \qquad (3)$$

with

$$\min paths(a) {=} \text{let } ml = \min(\bigcup_{x \in a} \|x\|)$$
$$\text{in} \bigcup_{x \in a} \text{ if } \|x\| = ml \text{ then } x \text{ else } \emptyset.$$

Here $path \min length$ select from a set of words the ones with the least number of letters. Therefore, we have the following algorithm for computing the shortest path between a set S and the node y:

$shortestpaths(S,y)$
$=$ {definition in Equ. 3}
$F(id, \min paths)(S,y)$
$=$ {Equ. 2}
if $\langle y \subseteq S\rangle \ \vee \ (S \bowtie R)\rangle \subseteq S\rangle$
then $\min paths(S \bowtie y)$
else $\min paths(S \bowtie y \cup shortestpaths(S \cup S \bowtie R, y))$fi
$=$ {Equ. 1}
if $y \in S\rangle$
then $\min paths(S \bowtie y)$
else if $(S \bowtie R)\rangle \subseteq S\rangle$
then $\min paths(S \bowtie y)$
else $\min paths(S \bowtie y \cup shortestpaths(S \cup S \bowtie R, y))$fi fi
$=$ $\{y \notin S\rangle \Rightarrow S \bowtie y = \emptyset\}$

$$
\begin{array}{l}
\text{if } y \in S\rangle \\
\quad \text{then } \min paths(S \bowtie y) \\
\quad \text{else if } (S \bowtie R)\rangle \subseteq S\rangle \\
\qquad \text{then } \min paths(\emptyset) \\
\qquad \text{else } \min paths(shortestpaths(S \cup S \bowtie R, y))\text{fi fi} \\
= \qquad \{\text{definition and idempotence of } \min paths\} \\
\text{if } y \in S\rangle \\
\quad \text{then } \min paths(S \bowtie y) \\
\quad \text{else if } (S \bowtie R)\rangle \subseteq S\rangle \\
\qquad \text{then } \emptyset \\
\qquad \text{else } shortestpaths(S \cup S \bowtie R, y))\text{fi fi.}
\end{array}
$$

References

[1] Tarski, A. : On the calculus of relations. J. Symb. Log. 1941,

[2] Schmidt, G. and Ströhlein, T. : Relations and graphs EATCS Monographs on Theoretical Computer Science. Springer - Verlag, Berlin, Heidelberg 1993.

[3] K. C. Ng. Relations with Transitive Closure. PhD thesis, University of California, Berkeley, 1984.

[4] K. C. Ng. and A. Tarski. Relation algebras with transitive closure, abstract 742-02-09. Notices Amer. Math. Soc., 24: A 29-A 30, 1977.

[5] Dexter Kozen. On induction vs. $*-$continuity. In Kozen, editor, Proc. Workshop on Logic of Programs, volume 131 of Lecture Notes in Computer Science, Page 167-176, New York,1981. Springer-Verlag.

[6] V. R. Pratt. Dynamic algebras as a well-behaved fragment of relation algebras. In D. Pigozzi, editor, Proc. Conf. on Algebra and Computer Science. Springer-Verlag, June 1988.

[7] Werner Kuich. The Kleene and Parikn theorem in complete semirings. In T. Ottmann, editor, Proc. 14th Colloq: Automata, Language, and Programming, volume 267 of Lecture Notes in Computer Science, page 212-225, New York, 1987. EATCS, Springer-Verlag.

[8] Werner Kuich and Arto Salomaa. Semirings, Automata, and Languages. Springer-Verlag, Berlin, 1986.

[9] Alfred V. Aho, John E. Hopcroft, and Jeffrey D. Ullman. The Design and Analysis of Computer Algorithms. Addison-Wesley, Reading, Mass., 1990.

[10] Kazuo Iwano and Kenneth Steiglitz. A semiring on convex polygons and zero-sum cycle problems. SIAM J. Comput., 19(5): 883-901, 1990.

[11] Dexter Kozen. The Design and Analysis of Algorithms. Springer-Verlag, New York, 1991.

[12] Kurt Mehlhorn. Data Structures and Algorithms 2: Graph Algorithms and NP-Completeness. EATCS Monographs on Theoretical Computer Science. Springer-Verlag, 1984.

[13] John Horton Conway. Regular Algebra and finite Machines. Chapman and Hall, London, 1971.

[14] Roland Carl Backhouse. Closure Algorithms and the star-Height Problem of Regular Languages. PhD thesis, Imperial College, London, U. K., 1975.

[15] S. Eilenberg. Automata, Languages, and Machines, volume A. Academic Press, New York, 1974.

[16] Annabelle Mclver. Mathematical foundations of Kleene Algebra. African Institute for Mathematical Sciences (AIMS). Stellenboch University, South Africa, (22 May 2008).

[17] T. Brunn, B. Moller, M. Russling. Layered Graph Traversals and Hamiltonian Path Problems - An Algebraic Approach. In: Jeuring, J. (ed.)MPC 1998. LNCS, vol.1422, pp.96-121. Springer, Heidelberg (1998).