Cuts of blocks of fuzzy relations and application in fuzzy databases

Vladimír Janiš[#1], Pavol Kral [*2], Magdalena Renčova[#3], Branimir Šešelja[♭4], Andreja Tepavčević[♭5]

[#] *Department of Mathematics, Faculty of Natural Sciences, Matej Bel University*
SK-974 01 Banská Bystrica, Slovak Republic
[1] vladimir.janis@umb.sk
[3] Magdalena.Rencova@umb.sk
[*] *Dept. of Quantitative Methods and Information Systems, Faculty of Economics*
Matej Bel University
SK-974 01 Banská Bystrica, Slovak Republic
[2] pavol.kral@umb.sk
[♭] *Department of Mathematics and Informatics, University of Novi Sad*
Trg Dositeja Obradovića 4, 21000 Novi Sad, Serbia
[4] seselja@dmi.uns.ac.rs
[5] andreja@dmi.uns.ac.rs

Abstract– The aim of this article is to present a construction of a fuzzy relation arising from a collection of data, which are supposed to be its blocks. We are motivated by managing databases and these block have properties usually appearing in some real systems. More precisely, we start with a collection of subsets closed under intersection and we want them to be cuts of blocks of a fuzzy relation, which can further be used for managing starting data. Then we give an algorithm for the construction of a lattice L and an L-fuzzy relation, such that the cuts of its fuzzy blocks are precisely the sets of the given collection. Further, we give conditions under which the obtained relation possesses some important relational properties. Finally, we analyze application of these results in fuzzy relational databases.

Key words and phrases– fuzzy relational databases, fuzzy relations, lattice-valued fuzzy set, lattice-valued fuzzy relation, block of fuzzy relation, cut, fuzzy similarity relation

I. Introduction

Databases that are presently mostly used are those which are based on relational models, and these are known as relational databases. Such tools are currently the predominant choice in storing different type of records, not only medical or financial ones, but also personnel data and many others. In all these databases, relation are usually constructed on a finite set of a fixed value domain. Every database is made of several relations and each record in a relation is a statement which can be "true" or "false". In other words, this classical model is based on binary approach. In order to grasp reality more closely, i.e., to handle grades of belongingness of elements to the relation, **fuzzy relational databases** have been developed in the near past. Recently these have become a widely investigated area (see [4], [12]). The first fuzzy relational database, FRDB, was described in [17]. Later, several other models appeared. In the context of fuzzy databases, some fuzzy querying languages have been constructed, that define some structures in order to include fuzzy aspects in the SQL statements (as fuzzy conditions, fuzzy comparators, fuzzy constants, fuzzy constraints, fuzzy thresholds and so on).

In the present paper we deal with the following problem. We start with a collection X of objects each of which is by some property connected to a group of other objects in X. Hence, there are as many subcollections as there are elements in X. Of course, subcollections are not necessarily disjoint and common objects in several of these also belong to a group determined by some object. Examples are numerous. X might consist of all members of a company or of an enterprize; each employe is professionally connected to a group of other members of the same company (e.g., belonging to a particular section, department, etc.). Another example could be a database in which items are classified according to their nature and properties: each item determines a sub-base.

Starting with the mentioned problem, we develop a procedure to construct a fuzzy relation out of the collection of subsets of a given sets, which are supposed to be the cuts of its blocks. Therefore, we are able to make a fuzzy relation for which the blocks are made of family of cuts we started from the beginning.

In the classical theory of relations, a block of a binary relation on a domain is its subset associated to an element from the domain: it contains all the elements from the domain related to the chosen one. The most widely used are blocks of equivalence relations which split the domain into equivalence classes. In databases, blocks of binary relations are known to be important and useful tool. Moreover, it is possible to reconstruct the relation if all the blocks are known.

We are here investigating fuzzy relations. In this case the situation is similar, although much more complicated. As block of the classical (crisp) relations are sets, blocks of a fuzzy relation are fuzzy sets on the same domain. For particular fuzzy relations, blocks have been widely investigated by many authors, sometimes under different names, (see e.g., [6], [9] and [1], [10], [11]) In the book [2] a comprehensive introduction to fuzzy (lattice valued) relations have been provided.

In paper [7] we developed an algorithm for constructing a relation out of its blocks. However, this algorithm is not applicable to all cases, and this disadvantage is resolved in the improved construction that is presented in this paper. In paper [8] we exploit resemblance relations in order to model some real life situations for which equivalences are not suitable. We study the properties of cuts for such relations. The construction of resemblance relation out of blocks of cuts was left as an open problem in the mentioned paper. The construction presented in the present paper can be applied also to this problem.

Following the motivation in databases framework presented above, here we deal with the following. We start with the collections of subsets of a set which is closed under intersections. Then we construct a lattice and a lattice valued relation such that the cuts of its blocks are starting sets. We were also able to find conditions which should be fulfilled by the starting closure system, in order that the constructed relation possesses some important relational properties: reflexivity, symmetry, transitivity and others.

The application of our results in managing databases is straightforward, due to the clear algorithm by which the fuzzy relation is constructed.

II. Preliminaries

Starting from a binary relation on a set X, $\rho \subseteq X^2$, for $x \in X$ we denote by $\rho[x]$ the subset of X defined by

$$\rho[x] := \{y \in X \mid (x,y) \in \rho\}. \tag{1}$$

The set $\rho[x]$ is called the **x-block** of ρ.

If the relation is intersection of other relations: $\rho = \bigcap_{i\in I} \rho_i$, then the block is the intersection of related blocks: $\rho[x] = \bigcap_{i\in I} \rho_i[x]$.

A **complete lattice** is a partially ordered set $(L, \leq)$ whose every subset has a least upper bound (join, supremum) and a greatest lower bound (meet, infimum) under $\leq$. A complete lattice has the top and the bottom element, denoted respectively by 1 and 0.

More details about lattices and related topics can be found e.g., in book [3].

As a generalization of a subset of a set, a **fuzzy set** $\mu : X \to L$ is a mapping from a non-empty set X (domain) into a complete lattice L (co-domain). According to the original definition L is the unit interval $[0,1]$ of real numbers (which is a complete lattice under $\leq$)[16]. Here we consider a complete lattice L in a more general setting and sometimes we use the term **L-fuzzy set**, or **lattice-valued** (**fuzzy**) **set** [5].

A mapping $R : X^2 \to L$ (a fuzzy set on X^2) is a **fuzzy** (**L-fuzzy, lattice-valued**) **relation** on X.

If $\mu : X \to L$ is a fuzzy set on a set X then for $p \in L$, the set

$$\mu_p := \{x \in X \mid \mu(x) \geq p\}$$

is a **p-cut**, or a **cut set**, (**cut**) of μ.

Fuzzy sets and fuzzy relations can be decomposed into their cuts and also they can be synthesized out of their cuts. More about this cutworthy approach to fuzzy structures can be found in e.g. [9], [13], [14], [15].

III. Results

We start from a complete lattice L and $R : X^2 \to L$, a fuzzy relation on a set X. For every $x \in X$, the **fuzzy x-block** (x-block) of R is the fuzzy set $R[x] : X \to L$, defined by

$$R[x](y) := R(x,y), \quad \text{for each } y \in X. \tag{2}$$

It should not lead to misunderstanding that we use the same name for both, the x-block of a usual and of a fuzzy relation; it should be clear from the context which of these notions is used. We refer to *blocks* of R, meaning x-blocks, where x runs over all $x \in X$. Since a block is a fuzzy set on X, then every x-block of R determines a collection of crisp subsets of X, its cuts.

In the opposite direction, we start from a family of subsets, and then we construct the corresponding fuzzy relation.

The following theorem givs conditions under which it is possible to construct such a relation.

Theorem 1: [7] *Let X be a nonempty set and for each $x \in X$ let $\mathcal{R}_x$ be a collection of nonempty subsets of X closed under set intersection and containing X. Then there is a lattice L and an L-valued relation R on X, such that for every $x \in X$, $\mathcal{R}_x$ is the collection of nonempty cuts of a relational block $R[x]$.*

In the following we describe a new construction of the lattice L and of the corresponding L-valued relation R (different from the one in [7]).

Let us recall the definition of the direct product and of the subdirect product.

Direct product:

$\prod_{i\in I} A_i$ is a set of all mappings p, satisfying the following: $p \in \prod_{i\in I} A_i$ if and only if p is a mapping $p : I \to \bigcup_{i\in I} A_i$ such that $p(i) \in A_i$ for every $i \in I$.

For $j \in I$, let $\pi_j : \prod_{i\in I} A_i \to A_j$, be **the j-th projection** defined by:

$$\pi_j(p) = p(j).$$

Subdirect product

Subdirect product is a subset of a direct product such that all the projections are onto mappings.

We start from the collection $\{\mathcal{R}_x \mid x \in X\}$, where for every $x \in X$,

$$\mathcal{R}_x = \{U_i^x \mid i \in I\}$$

is a family of subsets of X which is closed under set intersection and which contains X as a member.

Now we consider a subset $\mathcal{P}$ of the direct product $\prod_{x \in X} \mathcal{R}_x$, satisfying the following condition:

$$(\forall x)(\forall i)(\exists p \in \mathcal{P})(\exists y \in X) U_i^x = p(y).$$

In other words, $\mathcal{P}$ is a subdirect product of the collection $\{\mathcal{R}_x \mid x \in X\}$.

Now we define a family of relations $\{\rho_p \mid p \in \mathcal{P}\}$, as follows:

for every $p \in \mathcal{P}$,

$$(x, y) \in \rho_p \text{ if and only if } y \in p(x).$$

Now we consider a family of relations $\{\rho_p \mid p \in \mathcal{P}\}$ under inclusion, and add all the intersections of all the relations from the family, obtaining a lattice under the order dual to inclusion.

$$L := (\mathcal{C}, \leq),$$

where $\leq$ is the dual of set inclusion.

Finally, we define the relation $R : X^2 \to L$ by

$$R(x, y) := \bigcap \{\rho \in \mathcal{C} \mid (x, y) \in \rho\}.$$

We can prove that $R(x, y)$ is the required relation.

Next we characterize the properties of the obtained relation depending on the properties of the starting collections. We investigated reflexivity, irreflexivity, symmetry, antisymmetry and transitivity, since those are conditions that are usually required for relations. Using these properties we can directly obtain characterizations of fuzzy similarity (equivalence) relation and fuzzy ordering relation.

Theorem 2: If the starting collection $\{\mathcal{R}_x \mid x \in X\}$ fulfills the property that for every $x \in X$ and $y \in X$, we have that $y \in R_x$, then the relation R obtained by the construction above is reflexive.

Theorem 3: If the starting collection $\{\mathcal{R}_x \mid x \in X\}$ fulfills the property that no $x \in X$ is contained in a proper subset of X from R_x, then the relation R obtained by the construction above is irreflexive.

Theorem 4: Let $\{\mathcal{R}_x \mid x \in X\}$ be a starting collection. If there is a subdirect product $\mathcal{P} \subseteq \{\mathcal{R}_x \mid x \in X\}$ such that for every $p \in \mathcal{P}$ we have that from $x \in p(y)$ it follows that $y \in p(x)$, then the relation R obtained by the construction above is symmetric.

Theorem 5: Let $\{\mathcal{R}_x \mid x \in X\}$ be a starting collection. If there is a subdirect product $\mathcal{P} \subseteq \{\mathcal{R}_x \mid x \in X\}$ such that for every $p \in \mathcal{P}$ and $x \neq y$, from $x \in p(y)$ it follows that $y \notin p(x)$, then the relation R obtained by the construction above is antisymmetric.

Theorem 6: Let $\{\mathcal{R}_x \mid x \in X\}$ be a starting collection. If there is a subdirect product $\mathcal{P} \subseteq \{\mathcal{R}_x \mid x \in X\}$ such that for every $p \in \mathcal{P}$, from $x \in p(y)$ and $y \in p(z)$ it follows that $x \in p(z)$, the the relation R obtained by the construction above is transitive.

IV. Conclusion

The results that are presented above enable construction of fuzzy relation dealing with arbitrary data in a controlled and easy way. Namely, such a fuzzy relation shows connections among items, allow an easy handling and usage of these. In addition, it turns out that some important properties of the obtained fuzzy relation can be observed by looking at properties of the starting data.

Our aim is to further investigate constructions of fuzzy relations dealing with more complex structure of data.

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