Cut Approach to Closedness under Fuzzy Relations: Theoretical Basics and Applications in Artificial Intelligence

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Abstract. – The aim of this paper is to deal with the cutworthy approach to closedness of fuzzy sets under fuzzy relations. We present some theoretical results, and we prove that several important aspects of artificial intelligence and control problems can be successfully solved in this framework.

Keywords– fuzzy sets closed under fuzzy relations, fuzzy control

I. Introduction

Fuzzy IF-THEN rules appear in the context of fuzzy control problems, fuzzy approximate reasoning and fuzzy linear programming. In addition, each fuzzy controller is usually rule based: according to rules it acts on fuzzy input data and creates fuzzy output data. As it is known, the Mamdani approach to fuzzy controllers starts from a fuzzy relation which is deduced from an actual control process, and which from an input value creates an output value using a particular compositional rule of inference. In real world applications we often have a situation when output values are determined in advance by input values, and the problem is to find a fuzzy relation which performs such a transition.

What we prove here is that some of the foregoing problems arising in artificial intelligence could be analyzed, investigated and successfully solved in the framework of fuzzy sets closed under fuzzy relations.

Namely, we deal with the problem of finding an input fuzzy set μ, which will be closed under composition of a fuzzy relation, i.e. which satisfies the following property:

$$\bigvee_{x\in X}(\mu(x)\wedge R(x,y))\le \mu(y). \tag{1}$$

Using the properties of lattices, it is enough to find μ that is closed under relation R:

$$\mu(x)\wedge R(x,y)\le \mu(y).$$

Now, taking the supremum w.r.t. x, we obtain (1).

This type of formula is also used in fuzzy linear programming.

Here we solve (1) using cutworthy approach. We prove that the inequality is valid if and only if the analogue inequality is valid in all the cuts for different types of fuzzy (and boolean) sets and relations.

Further, we consider a special type of fuzzy relations. If R is a fuzzy similarity relation, then this approach has applications in approximate reasoning in the sense described in [8]. In case R is an ordering relation, fuzzy sets closed under fuzzy relations are applied in ordering based modifiers, as described in [2].

In the present research we deal mostly with transitive and similarity relations. Let us mention that in paper [5] we provided an investigation of ordering relation and related notions of fuzzy up-sets and down-sets in the context which can be considered as a starting point for the present research.

In organizing the paper for which the motivation is given above (Introduction), we briefly list some preliminary notions in the subsequent section (Preliminaries). Next we deal with necessary theoretical results (Section 3) which allow us to present the application (Section 4), as mentioned here.

II. Preliminaries

A. Relational systems; order

Let X be a non-empty set, $A \subseteq X$, a nonempty subset of X and $R \subseteq X^2$ a binary relation on X.

A subset A **is closed** with respect to relation R, if from $x \in A$ and $(x, y) \in R$ it follows that $y \in A$.

A **poset** is an ordered pair $(X, \leq)$ where X is a nonempty set and $\leq$ an ordering (reflexive, antisymmetric and transitive) relation on X. A **sub-poset** of a poset $(X, \leq)$ is a poset $(Y, \leq)$ where Y is a nonempty subset of X and $\leq$ on Y is a set-intersection of Y^2 and $\leq$ on X.

B. Fuzzy binary relations, fuzzy orderings

Let $\mu : X \to L$ be a fuzzy set, where $(L, \wedge, \vee, 0, 1)$ is a complete lattice. Then, for $p \in L$, a p**-cut** or cut μ_p is a subset of X, defined by:

$$\mu_p = \{x \in X \mid \mu(x) \geq p\}.$$

If $(L, \wedge, \vee, 0, 1)$ is a complete lattice, then $R : X^2 \to L$ is a fuzzy relation on X. A fuzzy relation on X is a fuzzy set on X^2.

A p**-cut** of a fuzzy relation on X is an ordinary relation on X: for $p \in L$,

$$R_p = \{(x, y) \in X^2 \mid R(x, y) \geq p\}.$$

Throughout the text, fuzzy sets (relations) are supposed to have L as the set of membership values.

(r) R is **reflexive** if for every $x \in X$, $R(x, x) = 1$.

(a) R is **antisymmetric** if for all $x, y \in X$,

if $x \neq y$ then either $R(x, y) = 0$ or $R(y, x) = 0$.

(t) R is **transitive** if for all $x, z, y \in X$, $R(x, y) \wedge R(y, z) \leq R(x, z)$.

(s) R is **symmetric** if for all $x, y \in X$, $R(x, y) = R(y, x)$.

Let us mention that there are other versions of the above properties, in particular if the lattice of values is residuated, or more specifically, defined on the real interval $[0, 1]$. Then T-norms are used instead of lattice operations. In addition, it is also possible to deal with fuzzy relations on fuzzy sets, and also to replace the crisp equality by the fuzzy one. For all these we refer to papers listed in References. We do not deal with such approaches here, since our framework allows us to present the most general results.

III. Some basic properties of fuzzy sets closed under fuzzy relations

Definition ([2]):

Let $\mu : X \to L$ be a fuzzy set and $R : X^2 \to L$ a fuzzy relation. Then μ is said to be **closed with respect to** R if for every $x, y \in X$

$$\mu(x) \wedge R(x, y) \leq \mu(y).$$

We can observe that this definition is a natural fuzzification of the analogous usual definition of a subset being closed with respect to an ordinary relation, mentioned in preliminaries. Namely, in meta-language we can exchange the conjunction with the minimum and the implication with the "less or equal" relation and then we obtain the definition of a fuzzy set closed with respect to a fuzzy relation.

In the following theorem we deal with fuzzy sets being closed w.r.t. a fuzzy relation. We prove that this property (closedness under a fuzzy relation) is a cutworthy one, i.e., that its crisp version is satisfied on all cuts if and only if it holds in the fuzzy case.

Theorem 1: *Let $\mu : X \to L$ be a fuzzy set and $R : X^2 \to L$ a fuzzy relation. The following are equivalent:*

(i) *μ is closed with respect to R.*

(ii) *for every $p \in L$, $p \neq 0$, the cut μ_p is closed with respect to (X, R_p).*

Next we state that any collection of subsets which is closed under set-intersections determine a fuzzy set whose family of cuts is the starting collection of subsets.

This proposition provides a tool for a construction of a fuzzy set closed under a fuzzy relation, using subsets that are closed under crisp relations.

Proposition 1: [7] *Let $\mathcal{F}$ be a family of subsets of a non-empty set X which is closed under intersections and contains X. Let $\alpha : X \to \mathcal{F}$ be defined by*

$$\alpha(x) = \bigcap(p \in \mathcal{F} \mid x \in p).$$

Then, α is an $\mathcal{F}$-valued set on X with the codomain lattice $(\mathcal{F}, \supseteq)$ such that its family of p-cuts is $\mathcal{F}$ and for every $p \in \mathcal{F}$ it holds that $p = \alpha_p$.

Obviously, one can start with the collection of subsets of X^2, and analogously as in Proposition 1 obtain a fuzzy relation whose family of cuts is the given collection. This fact is used in the following theorem.

Theorem 2: *Let $\mu : X \to L$ be a fuzzy set and $R : X^2 \to L$ a fuzzy relation. If μ is closed with respect to (X, R_p) for every $p \in L, p \neq 0$, then μ is closed with respect to R.*

The converse of the previous theorem is not satisfied, as shown by the following example.

Example 1: Let $X = \{a, b\}$ and let L be the complete lattice in Figure 1.

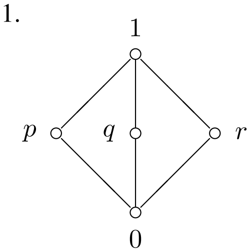

Figure 1.

Let $\mu : X \to L$ be defined by $\mu(a) = p$ and $\mu(b) = q$ and let R be a fuzzy relation (actually a fuzzy order) defined in the following table.

R	a	b
a	1	r
b	0	1

Table 1

It is easy to check that μ is closed under R (it is a fuzzy up-set on R), i.e., that the condition $\mu(x) \wedge R(x, y) \leq \mu(y)$ is valid for all $x, y \in X$.

Nevertheless, μ is not a fuzzy up-set on a crisp poset R_r (presented in Table 2), i.e., μ is not closed under the corresponding cut relations.

R	a	b
a	1	1
b	0	1

Table 2

Indeed, the condition $\mu(x) \wedge R_r(x, y) \leq \mu(y)$ is not valid for $x = a$ and $y = b$.

Hence we have illustrated that it is possible that μ is closed with respect to R (in this case to be a fuzzy up-set) and that μ is not closed with respect to (X, R_p) for every $p \in L, p \neq 0$ (in this case μ is not a fuzzy up-set on every crisp poset R_r).

Let us state the last result, in order to close the study of possible equivalences among crisp and fuzzy structures in this context.

Theorem 3: Let $\mu : X \to L$ be a fuzzy set and $R : X^2 \to L$ a fuzzy relation. The following are equivalent:

(iii) μ is closed with respect to (X, R_p) for every $p \in L, p \neq 0$;

(iv) for every $p \in L, p \neq 0$, the cut μ_p is closed with respect to the fuzzy relation R.

Remark 1: If we consider the notation for conditions used in Theorems 1 and 3, we have proven the following connections:

$$\begin{array}{ccc} (iii) & \longleftrightarrow & (iv) \\ & \big\downarrow & \\ (i) & \longleftrightarrow & (ii) \end{array}$$

The remained arrow is not included since it is not fulfilled. Therefore, it is stronger to ask whether a fuzzy set μ is closed with respect to any p-cut of R, than if it is closed with respect to the fuzzy relation R itself. Consequently, as witnessed by Theorems 1 and 3, the cutworthy approach is fulfilled in one direction only.

Theorem 4: The collection of all fuzzy sets closed with respect to a fuzzy relation R on a set X is a complete lattice under inclusion ($\subseteq$).

Next we consider fuzzy sets closed under relations fulfilling some of the mentioned properties.

Let $R : X^2 \to L$ be a fuzzy relation on X, and for $a \in X$, let $Ra : X \to L$ be the **class** of a under R, defined by $Ra(x) := R(a, x)$.

Proposition 1: R is a transitive fuzzy relation on X if and only if for every $a \in X$, the class Ra is closed under R.

Proposition 2: Let R be a transitive fuzzy relation on a finite set X, $A \subseteq X$ and let $\{Ra \mid a \in A\}$ be the family of the corresponding classes. Then:

(i) the fuzzy set $\bigcap_{a \in A} Ra$ is closed under R;

(ii) if the lattice L is distributive, then the fuzzy set $\bigcup_{a \in A} Ra$ is closed under R.

Proposition 3: If R is symmetric, then for a fuzzy set $\mu : X \to L$ closed under R we have that $\mu(x) \wedge R(x, y) = \mu(y) \wedge R(y, x)$ for all $x, y \in X$.

Proposition 4: If R is a fuzzy relation on X and for $a, b \in X$, $R(a, b) = R(b, a) = 1$, then every fuzzy set μ on X which is closed under R fulfils $\mu(a) = \mu(b)$.

The following lemma presents a known property of fuzzy equivalences.

Lemma 1: Let R be a fuzzy equivalence relation on X and $a, b \in X$. Then

$$R(a, b) = 1 \text{ if and only if } Ra = Rb.$$

Hence, we have the following result.

Corollary 1: Let R be an equivalence relation on X. If $Ra = Rb$ for some $a, b \in X$, then every fuzzy set μ on X closed under R fulfills $\mu(a) = \mu(b)$.

IV. Application: Fuzzy control problems

As we mentioned in Introduction, the above theoretical investigations could be suitably applied in some topics of artificial intelligence.

Namely, in fuzzy control and also in approximate reasoning it is often necessary to deal with fuzzy IF-THEN rules. In this context, each rule

$$\textit{IF } U \textit{ is } B \textit{ THEN } V \textit{ is } D$$

can be translated into a form

$$\textit{the pair } (B, D) \textit{ of } U \times V \textit{ takes the value in } R,$$

where R is a fuzzy relation.

Consequently, if U takes fuzzy input values from A, then we have

$$(U, V) \text{ is in relation } G, \text{ where } G = A \cap R.$$

Considering G and R as fuzzy relations on a universe X and A as a fuzzy set on the same universe, we obtain the following equality:

$$G(x, y) = A(x) \wedge R(x, y).$$

In this context, using the Mamdani approach to fuzzy controllers, we consider the following compositional rule of inference:

$$\mu_{A \circ R}(y) = \bigvee_{x \in X} (\mu(x) \wedge R(x, y)). \tag{2}$$

By the previous theoretical results, we are able to deal with the foregoing problems by analyzing the fuzzy case in the (usually much more simple) crisp environment. Namely, by the well known Synthesis theorem for fuzzy structures, the following is true.

$$\mu_{A \circ R}(y) = \bigvee_{p \in L} p \wedge (\mu_{A \circ R})_p(y), \tag{3}$$

where $(\mu_{A \circ R})_p$ stands for the characteristic function of the corresponding cut set, so that 0 and 1 from its co-domain are considered to be as well the bottom and the top of the lattice, respectively.

Therefore, we have the following practical statement.

Claim 1: Every fuzzy relation appearing in (2) is uniquely determined by its cut relations, and can be obtained by using the crisp objects only, as in Theorem 1, or by an application of combining methods, due to Theorems 2 and 3.

Observe that practical problems can require closedness under fuzzy relations with specific properties (like transitivity and others). Then the remaining results of the previous chapter can be appropriately applied.

V. Conclusion

Our paper deals with problems in fuzzy relational theory, which are motivated by some applications in artificial intelligence and control problems. Some aspects are successfully analyzed, though there are specific relational properties which appear in concrete applications, and still has to be investigated in the context of closedness under fuzzy relations.

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