n-Hosoya Polynomials of Saw and Thorn Saw Graphs

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Abstract

The n-Hosoya Polynomials of saw graph , and thorn saw graph are obtained . The Wiener indices and n-Wiener indices of these graphs are also determined .

Keywords: saw graph, thorn saw, n-Hosoya polynomial ,n-Wiener index.

1. Introduction:

We follow the terminology of [3],[4]. Let v be a vertex of a connected graph G , and let S be an (n-1)-subset of V(G) , $n \geq 2$, then the n-distance $d_n(v,S)$ is defined by [1]

$$d_n(v,S) = \min\{d(v,u) : u \in S\}. \qquad \ldots(1.1)$$

The n-diameter of G is defined by

$$diam_n G = \max\{d_n(v,S) : v \in V(G), |S| = n-1, S \subseteq V(G)\} \qquad \ldots(1.2)$$

The n-Wiener index of G is defined by

$$W_n(G) = \sum_{(v,S)} d_n(v,S). \qquad \ldots(1.3)$$

The n-Hosoya polynomial of connected graph G of order p is defined by

$$H_n(G;x) = \sum_{k=0}^{d_n} C_n(G,k)x^k, \qquad \ldots(1.4)$$

where $3 \leq n \leq p$, δ_n is the n-diameter of G , and $C_n(G,k)$ is the number of order pairs $(v,S), v \in V(G), S \subseteq V(G), |S| = n-1,$ such that $d_n(v,S) = k$.

One can easily show that [1].

$$C_n(G,0) = p\binom{p-1}{n-2},$$

$$C_n(G,1) = p\binom{p-1}{n-1} - \sum_{v \in V(G)} \binom{p-1-\deg v}{n-1} \ldots(1.5)$$

The n-Hosoya polynomial of a vertex v in G , denoted by $H_n(v,G;x)$, is defined [1] by

$$H_n(v,G;x) = \sum_{k \geq 0} C_n(v,G,k)x^k, \qquad \ldots(1.6)$$

where $C_n(v,G,k)$ is the number of (n-1)-subsets of vertices S such that $d_n(v,S) = k$. It is clear that for each k , $0 \leq k \leq d_n$,

$$C_n(G,k) = \sum_{v \in V(G)} C_n(v,G,k), \qquad \ldots(1.7)$$

and

$$H_n(G;x) = \sum_{v \in V(G)} H_n(v,G;x), \qquad \ldots(1.8)$$

The following simple lemma is useful for obtaining $C_n(v,G,k)$ for every vertex v of a connected graph G .

Lemma 1.1: [2] Let t be the number of vertices of ordinary distance k from vertex v , and let s be the number of vertices of distance more than k from v in a connected graph G . Then

$$C_n(v,G,k) = \binom{s+t}{n-1} - \binom{s}{n-1}, \qquad \ldots(1.9)$$

for $v \in V(G), 2 \leq n \leq p, 0 \leq k \leq d_n$. ■

Let T be a non-empty subset of vertices of G . We define

$$C_n(T,G,k) = \sum_{v \in T} C_n(v,G,k). \qquad \ldots(1.10)$$

We shall use this notation in our proofs.

In this paper , we obtain n-Hosoya polynomials and n-Wiener indices of saw and thorn saw graphs . Hosoya polynomials and Wiener indices are also determined in this paper .

2. Saw Graph:

A saw graph P_m^* is a path of order m, say $v_1, v_2, \dots, v_m$, with m-1 additional vertices $u_1, u_2, \dots, u_{m-1}$, and edges $\{u_i v_i, u_i v_{i+1} : i = 1,2,\dots,m-1\}$ as depicted in Fig .2.1.

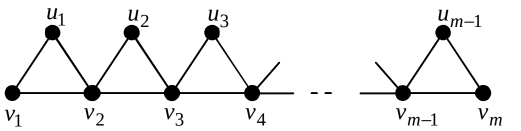

Fig. 2.1.

It is clear $p(P_m^*) = 2m-1$ and $q(P_m^*) = 3(m-1)$.

The n-diameter of P_m^* is given in the next statement.

Proposition 2.1: If P_m^* is a saw graph of order $p = 2m-1$, then

$$diam_n P_m^* = m - \left\lfloor \frac{n}{2} \right\rfloor, \quad 2 \le n \le p .$$

Proof: From Fig. 2.1 , one can easily notice that $diam_n P_m^* = e_n(v_1)$.

The n-eccentricity of v_1 is the n-distance from v_1 to the (n-1)-subset S consisting of the first n-1 vertices from the sequence :

$$v_m, u_{m-1}, v_{m-1}, u_{m-2} \dots, v_2, u_1 .$$

Thus , if n is even , then

$S = \{v_m; u_{m-1}, v_{m-1}; u_{m-2}, v_{m-2}; \dots; u_{m-(n/2-1)}, v_{m-(n/2-1)}\},\ n > 2$,

$S = \{v_m\},\ n = 2$,

and so $d_n(v_1, S) = m - \frac{n}{2}$.

If n is odd , then

$S = \{v_m, u_{m-1}; v_{m-1}, u_{m-2}; \dots; v_{m-((n-1)/2-1)}, u_{m-(n-1)/2)}\}$ and so $d_n(v_1, S) = m - \frac{n-1}{2}$.

Hence , the proof. ■

To find $H_n(P_m^*; x)$, we redraw P_m^*, and relabel its vertices as shown in Fig.2.2

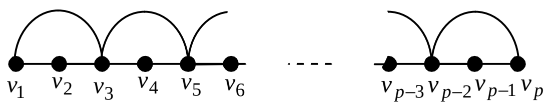

Fig.2.2 The saw graph P_m^* where p=2m-1

Theorem 2.2: For $2 \le k \le d_n$, the coefficient $C_n(P_m^*, k)$ of the n-Hosoya polynomial of the saw graph P_m^* of order $p = 2m-1$, $m \ge 4$, is given by

$$C_n(P_m^*, k) = 4\binom{a+2k+1}{n-1} + R ,$$

$$R = \begin{cases} \frac{a+3}{2}\binom{a+5}{n-1} + \frac{a-3}{2}\binom{a+3}{n-1} - \frac{a+7}{2}\binom{a+1}{n-1} \\ \qquad - \frac{a+1}{2}\binom{a-1}{n-1}; \ if\ 2 \le k \le \left\lfloor \frac{m}{2} \right\rfloor - 1 \\ zero \qquad ; if\ \left\lfloor \frac{m}{2} \right\rfloor + 1 \le k \le d_n \end{cases}$$

and

$$C_n(P_m^*, k) = 4\binom{a+2k+1}{n-1} + \binom{4}{n-1} - 2\binom{a+3}{n-1} - 2\binom{a+1}{n-1} + \overline{R} ; \ if\ k = \left\lfloor \frac{m}{2} \right\rfloor$$

$$\overline{R} = \begin{cases} zero & ; \ if\ m\ is\ even \\ 2\binom{6}{n-1} - 2\binom{2}{n-1} & ; \ if\ m\ is\ odd \end{cases}$$

Where $a = p - 4k$, and δ_n is the n-diameter of P_m^* .

Proof: Let v_i, $1 \le i \le p$, and S be an (n-1)-subset of vertices. We consider two cases for the values of k .

<u>Case (1)</u>: When $2 \le k \le \left\lfloor \frac{m}{2} \right\rfloor$.

For $1 \le i \le 2k-1$, there are exactly two vertices, namely $v_{2k+2\lfloor (i-1)/2 \rfloor}$ and $v_{2k+2\lfloor (i-1)/2 \rfloor+1}$, of distance k from v_i, and there are $p-2k-2\lfloor (i-1)/2 \rfloor -1$ vertices of distance more than k from v_i.

Hence for such values of i , the number of (n-1)-subsets such that $d_n(v_i, S) = k$ is given by

$$\sum_{j=1}^{n-1} \binom{2}{j} \binom{p-2k-2\lfloor (i-1)/2 \rfloor -1}{n-j-1}, \text{for } 1 \le i \le 2k-1$$

Hence

$$\sum_{i=1}^{2k-1} C_n(v_i, P_m^*, k) = \sum_{i=1}^{2k-1} \left[\binom{p-2k-2\lfloor (i-1)/2 \rfloor +1}{n-1} - \binom{p-2k-2\lfloor (i-1)/2 \rfloor -1}{n-1} \right]$$

$$= \sum_{i=1}^{2k-2} \left[\binom{p-2k-2\lfloor (i-1)/2 \rfloor +1}{n-1} - \binom{p-2k-2\lfloor (i-1)/2 \rfloor -1}{n-1} \right] + \left[\binom{p-4k+3}{n-1} - \binom{p-4k+1}{n-1} \right]$$

$$= 2\sum_{i=k}^{2k-2} \left[\binom{p-2i+1}{n-1} - \binom{p-2i-1}{n-1} \right] + \left[\binom{a+3}{n-1} - \binom{a+1}{n-1} \right]$$

$$= 2\left[\binom{p-2k+1}{n-1} - \binom{a+3}{n-1} \right] + \left[\binom{a+3}{n-1} - \binom{a+1}{n-1} \right]$$

$$= 2\binom{a+2k+1}{n-1} - \binom{a+3}{n-1} - \binom{a+1}{n-1}. \qquad \ldots(2.1)$$

Because of the symmetry of P_m^* with respect to the initial vertex v_1 and the terminal vertex v_p, the number of (n-1)-subsets S such that $d_n(v_i, S) = k$, for $p-2k+2 \le i \le p$, is given by (2.1)

Hence for this case and for the values of i, $1 \le i \le 2k-1$ and $p-2k+2 \le i \le p$, we have

$$\sum_i C_n(v_i, P_m^*, k) = 4\binom{a+2k+1}{n-1} - 2\binom{a+3}{n-1} - 2\binom{a+1}{n-1} \qquad \ldots(2.2)$$

Now , for $2k \le i \le p-2k+1$, and $2 \le k \le \left\lfloor \frac{m}{2} \right\rfloor -1$, there are four vertices of distance k from v_i, namely , $v_{2\lfloor i/2 \rfloor -2k+1}, v_{2\lfloor i/2 \rfloor -2k+2}, v_{2\lceil i/2 \rceil +2k-2}$, and $v_{2\lceil i/2 \rceil +2k-1}$. The number of vertices in P_m^* of distance more than k from v_i is given by

$$p - \{(2\lceil i/2 \rceil + 2k-1) - (2\lfloor i/2 \rfloor - 2k+1) + 1\}$$
$$= a - 2h_i + 1$$

where $h_i = \begin{cases} 0 \text{ , when } i \text{ is even} \\ 1 \text{ , when } i \text{ is odd} \end{cases}$

Hence for this case , the number of pairs (v_i, S) of n-distance k and for all i, $2k \le i \le p-2k+1$, is given by

$$\sum_{i=2k}^{p-2k+1} \sum_{j=1}^{n-1} \binom{4}{j} \binom{a-2h_i+1}{n-1-j}. \qquad \ldots(2.3)$$

It can easily be checked that when p is odd , then the number of even i's is $\frac{a+3}{2}$, and the number of odd i's is $\frac{a+1}{2}$.

Because, p is always odd for P_m^*, then (2.3) simplified to

$$\frac{a+3}{2}\sum_{j=1}^{n-1}\binom{4}{j}\binom{a+1}{n-1-j}+\frac{a+1}{2}\sum_{j=1}^{4}\binom{4}{j}\binom{a-1}{n-1-j}$$

$$=\frac{a+3}{2}\left[\binom{a+5}{n-1}-\binom{a+1}{n-1}\right]+\frac{a+1}{2}\left[\binom{a+3}{n-1}-\binom{a-1}{n-1}\right] \quad \ldots.(2.4)$$

Finally , we note from Fig.2.2 , that

$$C_n(v_m,P_m^*,\left\lfloor\frac{m}{2}\right\rfloor)=\binom{4}{n-1}, \text{ if } m \text{ is even or odd} , \quad \ldots(2.5)$$

and

$$C_n(v_{m-1},P_m^*,\left\lfloor\frac{m}{2}\right\rfloor)=C_n(v_{m+1},P_m^*,\left\lfloor\frac{m}{2}\right\rfloor)$$

$$=\sum_{j=1}^{n-1}\binom{4}{j}\binom{2}{n-1-j}$$

$$=\binom{6}{n-1}-\binom{2}{n-1}, \text{ if } m \text{ is odd} \quad \ldots.(2.6)$$

From above , (2.2) , (2.4), (2.5), and (2.6), we have

$$C_n(P_m^*,k)=4\binom{a+2k+1}{n-1}+y , \text{ where}$$

$$y=\begin{cases}\frac{a+3}{2}\binom{a+5}{n-1}+\frac{a-3}{2}\binom{a+3}{n-1}-\frac{a+7}{2}\binom{a+1}{n-1} \\ \qquad -\frac{a+1}{2}\binom{a-1}{n-1}; if\ 2\le k\le\left\lfloor\frac{m}{2}\right\rfloor-1 \\ \binom{4}{n-1}-2\binom{a+3}{n-1}-2\binom{a+1}{n-1}+j\ ; if\ \ k=\left\lfloor\frac{m}{2}\right\rfloor\end{cases}$$

where

$$j=\begin{cases}zero & ; if\ \ m \text{ is even} \\ 2\binom{6}{n-1}-2\binom{2}{n-1} & ; if\ \ m \text{ is odd}\end{cases} \quad \ldots.(\ 2.7)$$

<u>Case (2)</u>: When $\left\lfloor\frac{m}{2}\right\rfloor+1\le k\le d_n$.

For $1\le i\le 2(m-k)$, then we have the same result of Case (1) for $1\le i\le 2k-1$. That is:

$$C_n(v_i,P_m^*,k)=\sum_{j=1}^{n-1}\binom{2}{j}\binom{p-2k-2\lfloor(i-1)/2\rfloor-1}{n-1-j},$$

$1\le i\le 2(m-k)$.

Hence

$$\sum_{i=1}^{2(m-k)}C_n(v_i,P_m^*,k)=\sum_{i=1}^{2(m-k)}\left[\binom{p-2k-2\lfloor(i-1)/2\rfloor+1}{n-1}-\binom{p-2k-2\lfloor(i-1)/2\rfloor-1}{n-1}\right]$$

$$=2\left[\binom{a+2k+1}{n-1}-\binom{0}{n-1}\right]=2\binom{a+2k+1}{n-1} \quad \ldots.(2.8)$$

Also because of the symmetry of P_m^* , the number of (n-1)-subsets S such that $d_n(v_i,S)=k$, for $2k\le i\le p$ is given by (2.8).

Hence for this case of the values of i , $1\le i\le 2(m-k)$ and $2k\le i\le p$, we have

$$\sum_i C_n(v_i,P_m^*,k)=4\binom{a+2k+1}{n-1}$$

This complete the proof . ■

<u>Corollary 2.3:</u> The n-Hosoya polynomial of saw graph P_m^* of order p=2m-1 is given by

$$H_n(P_m^*;x)=p\binom{p-1}{n-2}+\left\{p\binom{p-1}{n-1}-\frac{p+3}{2}\binom{p-3}{n-1}-\frac{p-3}{2}\binom{p-5}{n-1}\right\}x+\sum_{k=2}^{d_n}C_n(P_p^*,k)\,x^k$$

, for $m\ge 4$,

$$H_n(P_2^*;x)=3\binom{2}{n-2}+3\binom{2}{n-1}x$$

$$H_n(P_3^*;x)=5\binom{4}{n-2}+\left\{5\binom{4}{n-1}-4\binom{2}{n-1}\right\}x+4\binom{2}{n-1}x^2 ,$$

and the n-Wiener index of saw graph P_m^* is

$$W_n(P_p^*) = p\binom{p-1}{n-1} - \frac{p+3}{2}\binom{p-3}{n-1} - \frac{p-3}{2}\binom{p-5}{n-1} + \sum_{k=2}^{d_n} kC_n(P_p^*,k),$$

$$W_n(P_2^*) = 3\binom{2}{n-1},$$

$$\text{and } W_n(P_3^*) = 5\binom{4}{n-1} + 4\binom{2}{n-1},$$

where $C_n(P_p^*,k)$ is given in Theorem 2.2, for $2 \le k \le d_n$, and δ_n is the n-diameter determined by Proposition 2.1.

Corollary 2.4: The Hosoya polynomial of saw graph P_m^* of order p=2m-1 is given by

$$H(P_m^*;x) = (2m-1) + 3(m-1)x + 4\sum_{i=2}^{m-1}(m-i)x^i . \quad \ldots(2.9)$$

Proof: When n=2 , we have $d_2(v,\{u\}) = d_2(u,\{v\})$.

Thus $H(P_m^*;x)$ is obtained from Theorem 2.2 , by putting n=2 , and dividing by 2.

We notice that our result , for special value of n=2 is exactly the result obtained by W. A.M. Saeed [5].

From (2.9) we notice that: $C(P_m^*,0) = 2m-1$, $C(P_m^*,1) = 3(m-1)$, and $C(P_m^*,k) = 4(m-k)$, for $2 \le k \le m-1$.

Therefore

$C(P_m^*,0) < C(P_m^*,1) < C(P_m^*,2) > C(P_m^*,3) >$
$C(P_m^*,4) > \ldots > C(P_m^*,d)$

Thus , the sequence $(C(P_m^*,k))$ is strong – unimodal, and clearly , it is neither palindromic nor semi-palindromic .

3. Thorn Saw Graphs:

Definition 3.1: Let G be a connected graph of order p and maximum degree Δ. **A n-thorny graph** G^* is the graph constructed from G be adding pendals such that every vertex of G becomes of degree n , where $\Delta \le n$, in G^*.

In most chemical graphs we have $\Delta \le 4$ and we take $n = 4$ in constructing n- thorny graphs. For saw graphs P_m^* of order $2m-1, m \ge 2$, we take $n = 4$. Then , the thorn saw P_m^{*c} of order $p = 4m+1, m \ge 2$, is shown in Fig. 3.1.

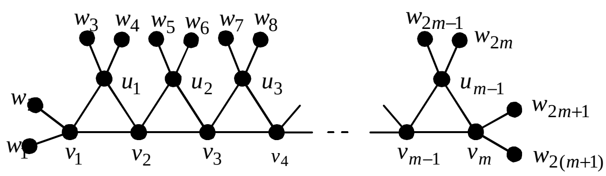

Fig. 3.1. Thorn saw graph P_m^{*c} of order $p = 4m+1$

The number of added end-vertices is $2(m+1)$ and labeled $w_1, w_2, \ldots, w_{2m+2}$. It is clear that $q(P_m^{*c}) = 5m-1$, and the diameter of P_m^{*c} is $m+1$, it is $d(w_1, w_{2m+2})$. The first result determines the n-diameter of P_m^{*c}.

Proposition3.1: The n-diameter of the thorn saw of order $p = 4m+1, \quad m \ge 2$, is given by

$$diam_n P_m^{*c} = \begin{cases} m+1-\left\lfloor \frac{n-2}{4} \right\rfloor, & 2 \le n \le 4m \\ 1 & , \ n = 4m+1 \end{cases}$$

Proof: It is clear from Fig.3.1 , that

$diam_n P_m^{*c} = e_n(w_1).$

The n-eccentricity of w_1 is the n-distance from w_1 to the (n-1)-subset S consisting of the first n-1 vertices from the sequence :

$w_{2m+2}, w_{2m+1}, w_{2m}, w_{2m-1}; v_m, u_{m-1}, w_{2m-2}, w_{2m-3}$ $; v_{m-1}, u_{m-2}, w_{2m-4}, w_{2m-5}; \ldots ; v_3, u_2, w_4, w_3;$ $v_2, u_1, w_2; v_1$.

Notices that distance from w_1 to each of the four vertices $v_j, u_{j-1}, w_{2j-2}, w_{2j-3}$, for $3 \le j \le m$, is j . Also, each vertex of $\{w_{2m+2}, w_{2m+1}, w_{2m}, w_{2m-1}\}$ is of distance $m+1$ from w_1, and if $n \ge 5$, then S must contain all these vertices. Therefore, we have $4(m+1-j) = n-5$, which implies $4j = 4(m+2)-(n-1)$.

Since j is integer , then

$$j = m+2-\left\lceil \frac{n-1}{4} \right\rceil = m+1-\left\lfloor \frac{n-2}{4} \right\rfloor.$$

For $n = 4m-h, \ 0 \le h \le 2, \ d_n(w_1, S) = 2$, which is the same result obtained from Fig. 3.1. ■

In the next theorem, we find the n-Hosoya Polynomial of P_m^{*c} .

To simplify the proofs we redraw P_m^{*c}, and relabel its vertices as shown in Fig .3.2 . It is clear that $p = 4m+1$.

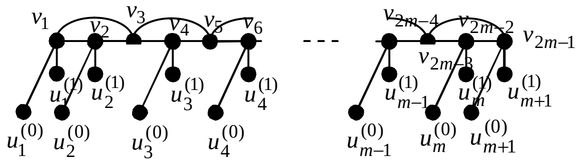

Fig.3.2.

Proposition3.2: For $3 \le n \le 4m+1$, and $v_{j+1} \ for \ j = 0,1$,

$$C_n(P_m^{*c}, 2) = 2(m+1)\binom{4m-1}{n-1} - (m+1)\binom{4m-8}{n-1} - (m-2)\binom{4m-12}{n-1} - 3\binom{4m-4}{n-1}. \quad \ldots(3.1)$$

Proof: Let S be an (n-1)-subset of $V(P_m^{*c})$. We consider three cases for a vertex $w \in V(P_m^{*c})$.

<u>**Case(1):**</u> $w = u_i^{(r)}$, $r = 0,1$ and $i = 1,2, \ldots, m+1$

There are three vertices, namely $u_1^{(1-r)}, v_2, v_3$ each of distance 2 from $u_1^{(r)}$ $r = 0,1$ .Also, there are three vertices, namely $u_{m+1}^{(1-r)}, v_{2m-2}, v_{2m-3}$ each of distance 2 from $u_{m+1}^{(r)}$, $r = 0,1$. Moreover, each of the three vertices $u_i^{(1-r)}, v_{2i-3}, v_{2i-1}$ is of distance 2 from $u_i^{(r)}$ $for \ i = 2,3, \ldots, m \ and \ r = 0,1$.

Therefore, the number of pairs $(u_i^{(r)}, S)$ such that $d_n(u_i^{(r)}, S) = 2, \quad i = 1,2, \ldots, m+1$, and $r = 0,1$ is given by

$$2(m+1)\sum_{j=1}^{n-1}\binom{3}{j}\binom{p-5}{n-1-j} = 2(m+1)\left[\binom{4m-1}{n-1} - \binom{4m-4}{n-1}\right] \quad \ldots.(3.2)$$

Case (2): $w = v_1, v_2; v_4; \ldots ; v_{2m-4}; v_{2m-2}, v_{2m-1}$.

There are four vertices, namely $v_4, v_5, u^{(0)}_{2-j}, u^{(1)}_{2-j}$ each of distance 2 from v_{j+1} *for* $j = 0,1$. Also there are four vertices, namely $v_{2m-5}, v_{2m-4}, u^{(0)}_{m+j}, u^{(1)}_{m+j}$ each of distance 2 from v_{2m-1-j}, *for* $j = 0,1$.

Moreover, each of the four vertices $v_{2i-3}, v_{2i-2}, v_{2i+2}, v_{2i+3}$, is of distance 2 from vertex v_{2i} *for* $i = 2,3, \ldots , m-2$.

Therefore the number of pairs (w,S) such that $d_n(w,S) = 2$, is given by

$$(m+1)\sum_{j=1}^{n-1}\binom{4}{j}\binom{p-9}{n-1-j} = (m+1)\left[\binom{4m-4}{n-1} - \binom{4m-8}{n-1}\right] \quad \ldots.(3.3)$$

Case (3): $w = v_3, v_5, \ldots , v_{2m-5}, v_{2m-3}$. There are eight vertices, namely $u^{(0)}_1, u^{(1)}_1, u^{(0)}_2, u^{(1)}_2, u^{(0)}_3, u^{(1)}_3, v_6, v_7$ each of distance 2 from v_3. Also there are eight vertices, namely

$u^{(0)}_{m+1}, u^{(1)}_{m+1}, u^{(0)}_m, u^{(1)}_m, u^{(0)}_{m-1}, u^{(1)}_{m-1}, v_{2m-6}, v_{2m-7}$

each of distance 2 from v_{2m-3}.

Moreover, each of the eight vertices $v_{2i-1}, v_{2i}, u^{(0)}_{i+2}, u^{(1)}_{i+2}, u^{(0)}_{i+3}, u^{(1)}_{i+3}, v_{2i+6}, v_{2i+7}$ is of distance 2 from vertex v_{2i+3} for $i = 1,2, \ldots , m-4$.

Therefore, the number of pairs (w,S) such that $d_n(w,S) = 2$, is given by

$$(m-2)\sum_{j=1}^{n-1}\binom{8}{j}\binom{p-13}{n-1-j} = (m-2)\left[\binom{4m-4}{n-1} - \binom{4m-12}{n-1}\right] \quad \ldots.(3.4)$$

Hence, from (3.2), (3.3), and (3.4) we obtain (3.1).

Remark 1: If m=2, then Case (3) does not exist and so

$$C_n(P_m^{*c}, 2) = 6\binom{7}{n-1} - 3\binom{4}{n-1}.$$

Proposition 3.3: For $3 \le n \le 4m+1$, and $m \ge 5$,

$$C_n(P_m^{*c}, 3) = 2(m+1)\binom{4m-4}{n-1} + (m-6)\binom{4m-12}{n-1} - (m+1)\binom{4m-8}{n-1} - (m-1)\binom{4m-16}{n-1} - (m-4)\binom{4m-20}{n-1}. \quad \ldots(3.5)$$

Proof: We have two cases:

Case (1): $w \in A \cup B$, where

$A = \{u^{(r)}_i : r = 0,1 \text{ and } i = 1,2, \ldots , m+1\}$

$B = \{v_i : i = 1,2,3,2m-1,2m-2,2m-3\}$.

As in the proof of Proposition 3.2, one can easily show that there are four vertices of distance 3 from vertex $w \in A \cup B$; and there are p-9 vertices of distance more than 3 when $w \in A$; there are p-13 vertices of distance more than 3 when $w \in \{v_1, v_2, v_{2m-1}, v_{2m-2}\}$; and there are p-17 vertices of distance more than 3 when $w \in \{v_3, v_{2m-3}\}$.

Therefore, the number of pairs (w,S) such that $d_n(w,S) = 3$, is given by

$$2(m+1)\sum_{j=1}^{n-1}\binom{4}{j}\binom{p-9}{n-1-j} + 4\sum_{j=1}^{n-1}\binom{4}{j}\binom{p-13}{n-1-j} + 2\sum_{j=1}^{n-1}\binom{4}{j}\binom{p-17}{n-1-j}$$

$$= 2(m+1)\left[\binom{4m-4}{n-1} - \binom{4m-8}{n-1}\right] + 4\left[\binom{4m-8}{n-1} - \binom{4m-12}{n-1}\right] + 2\left[\binom{4m-12}{n-1} - \binom{4m-16}{n-1}\right]$$

$$= 2(m+1)\binom{4m-4}{n-1} - 2(m-1)\binom{4m-8}{n-1} - 2\binom{4m-12}{n-1} - 2\binom{4m-16}{n-1}. \quad \ldots(3.6)$$

<u>Case (2)</u>: $w \in D = \{v_i : \ i = 4, 5, \ldots, 2m-4\}$.

For each such $w \in D$ there are eight vertices each of distance 3 from w. But, there are (p-17) vertices of distance more than 3 from vertex v_i *for even* i, $4 \le i \le 2m-4$; and there are (p-21) vertices of distance more than 3 from v_i *for odd* i, $5 \le i \le 2m-5$.

Therefore, the number of pairs (w, S), $w \in D$, such that $d_n(w, S) = 3$, is given by

$$(m-3)\sum_{j=1}^{n-1}\binom{8}{j}\binom{p-17}{n-1-j} + (m-4)\sum_{j=1}^{n-1}\binom{8}{j}\binom{p-21}{n-1-j}$$
$$= (m-3)\binom{4m-8}{n-1} + (m-4)\binom{4m-12}{n-1} - (m-3)\binom{4m-16}{n-1} - (m-4)\binom{4m-20}{n-1}. \quad \ldots(3.7)$$

Hence from (3.6) and (3.7), we obtain (3.5). ■

Remark 2:

$$C_n(P_2^{*c}, 3) = 6\binom{4}{n-1},$$

$$C_n(P_3^{*c}, 3) = 8\binom{8}{n-1} - 4\binom{4}{n-1},$$

$$C_n(P_4^{*c}, 3) = 10\binom{12}{n-1} - 5\binom{8}{n-1} - 2\binom{4}{n-1}.$$

Proposition 3.4: For $3 \le n \le p = 4m+1$, $4 \le k \le \left\lceil \frac{m}{2} \right\rceil$, $m \ge 7$, the coefficient of x^k in the n-Hosoya polynomial of P_m^{*c} is given by

$$C_n(P_m^{*c}, k) = 8\binom{a+7}{n-1} + (2m+10-4k)\binom{a+23-4k}{n-1} + (m-2k)\binom{a+11-4k}{n-1} - 4\binom{a+19-4k}{n-1} - (m+7-2k)\binom{a+15-4k}{n-1} - (m+5-2k)\binom{a+7-4k}{n-1} - (m+2-2k) - \binom{a+3-4k}{n-1}, \quad \ldots(3.8)$$

where $a = p - 4k$, *and* $p = 4m+1$.

Proof: We consider two cases and partition $V(P_m^{*c})$ in to A and B in which each vertex w of A is of distance k from exactly four vertices; and each vertex w' of B is of distance k from exactly eight vertices.

<u>Case (1):</u> $A = A_1 \cup A_2 \cup A_2' \cup A_3 \cup A_3' \cup A_4$, where

$A_1 = \{u_i^{(r)} : r = 0, 1; i = 1, 2, m+1, m\}$,

$A_2 = \{u_i^{(r)} : r = 0, 1; i = 3, 4, \ldots, k-2; \ k \ne 4\}$,

$A_2' = \{u_i^{(r)} : r = 0, 1; i = m-1, m-2, \ldots, m-k+4; \ k \ne 4\}$,

$A_3 = \{v_i : \ i = 1, 2, \ldots, 2k-5, 2k-4\}$,

$A_3' = \{v_i : \ i = 2m-1, 2m-2, \ldots, 2m-2k+4\}$,

$A_4 = \{v_{2k-3}, v_{2m-2k+3}\}$.

For each $w \in A$, there are four vertices each of distance k from w. For instance, each of vertices $v_{2k}, v_{2k+1}, u_k^{(0)}, u_k^{(1)}$ is of distance k from vertex v_i, $i = 1,2$. Also, each of

$v_{4k-10}, v_{4k-9}, u^{(0)}_{2k-5}, u^{(1)}_{2k-5}$ is of distance k from $u^{(r)}_{k-2}$, $r = 0,1$.

If $w \in A_1$, then there are p-4k+3 vertices each of distance more than k from w. Thus

$$C_n(w, P_m^{*c}; k) = \sum_{j-1}^{n-1} \binom{4}{j}\binom{a+3}{n-1-j}, \ \ for\ w \in A_1 .$$

…(3.9)

If $w \in A_2$, say $w = u_i^{(r)}$, then there are p-4k+11-4i vertices each of distance more than k from $u_i^{(r)}$. Thus

$$C_n(u_i^{(r)}, P_m^{*c}; k) = \sum_{j-1}^{n-1} \binom{4}{j}\binom{a+11-4i}{n-1-j}, \ \ for\ u_i^{(r)} \in A_2$$

From Fig.3.2, one may notice that

$C_n(u_i^{(r)}, P_m^{*c}, k) = C_n(u^{(r)}_{m-i+2}, P_m^{*c}, k)$,

$i = 3,4, \ \ldots \ , k-2$, $r = 0,1$, and therefore the number of pairs (w,S), if $w \in A_2 \cup A_2'$, such that $d_n(w,S) = k$, is given by:

$$2\sum_{i=3}^{k-2} \sum_{j-1}^{n-1} \binom{4}{j}\binom{a+11-4i}{n-1-j} = 2\sum_{i=3}^{k-2}\left[\binom{a+15-4i}{n-1} - \binom{a+11-4i}{n-1}\right]$$

$$= 4\left[\binom{a+3}{n-1} - \binom{a+19-4k}{n-1}\right]. \qquad \ldots(3.10)$$

If $w \in A_3$ say $w = v_i$, then there are $p-4k+3-4\left\lceil \frac{i}{2} \right\rceil$ vertices each of distance more than k from v_i. Thus

$$C_n(v_i, P_m^{*c}, k) = \sum_{j-1}^{n-1} \binom{4}{j}\binom{a+3-4\lceil i/2\rceil}{n-1-j}, for\ v_i \in A_3 .$$

Also , from Fig.3.2, one may notice that

$C_n(v_i, P_m^{*c}, k) = C_n(v_{2m-i}, P_m^{*c}, k)$, $i = 1,2, \ \ldots \ , 2k-4$,and therefore the number of pairs (w,S), $w \in A_3 \cup A_3'$, such that $d_n(w,S) = k$ is given by:

$$2\sum_{i=1}^{2k-4} \sum_{j-1}^{n-1} \binom{4}{j}\binom{a+3-4\lceil i/2\rceil}{n-1-j} = 2\sum_{i=1}^{2k-4}\left[\binom{a+7-4\lceil i/2\rceil}{n-1} - \binom{a+3-4\lceil i/2\rceil}{n-1}\right]$$

$$= 4\sum_{i=1}^{k-2}\left[\binom{a+7-4i}{n-1} - \binom{a+3-4i}{n-1}\right]$$

$$= 4\left[\binom{a+3}{n-1} - \binom{a+11-4k}{n-1}\right] \ldots.(3.11)$$

If $w \in A_4$,then there are p-8k+7 vertices each of distance more than k from w. Thus

$$C_n(w, P_m^{*c}, k) = \sum_{j-1}^{n-1} \binom{4}{j}\binom{a+7-4k}{n-j-1}, \ \ for\ \ w \in A_4$$

….(3.12)

Therefore, from (3.9)-(3.12), we get the number of pairs (w,S), $w \in A$, such that $d_n(w,S) = k$, which is

$$8\left[\binom{a+7}{n-1} - \binom{a+3}{n-1}\right] + 4\left[\binom{a+3}{n-1} - \binom{a+19-4k}{n-1}\right]$$
$$+4\left[\binom{a+3}{n-1} - \binom{a+11-4k}{n-1}\right] + 2\left[\binom{a+11-4k}{n-1} - \binom{a+7-4k}{n-1}\right]$$

….(3.13)

<u>Case (2):</u> $B = B_1 \cup B_2 \cup B_3$, where

$B_1 = \{u_i^{(r)} : r = 0\,,1 \ and\ i = k-1, k, k+1, \ \ldots \ , m+3-k\}$,

$B_2 = \{v_i : \ i = 2k-2, 2k \ ; 2k+2, \ \ldots \ , 2m-2k+2\}$,

$B_3 = \{v_i : \ i = 2k-1, 2k+1 \ ; 2k+3, \ \ldots \ , 2m-2k+1\}$

For each $w' \in B$, there are eight vertices each of distance k from w'. For instance, vertex $u^{(r)}_{k-1}, (r = 0,1)$ is of distance k from each of the eight vertices

$u_1^{(0)}, u_1^{(1)}, u_2^{(0)}, u_2^{(1)}, u_{2k-4}^{(0)}, u_{2k-4}^{(1)}, v_{4k-7}, v_{4k-8}$.

If $w' \in B_1$, then there are p-8k+15 vertices each of distance more than k from w'. Thus

$$C_n(w', P_m^{*c}, k) = \sum_{j=1}^{n-1} \binom{8}{j}\binom{a+15-4k}{n-1-j}, \ for\ w' \in B_1 \quad \text{....(3.14)}$$

If $w' \in B_2$, then there are p-8k+7 vertices each of distance more than k from w'. Thus

$$C_n(w, P_m^{*c}, k) = \sum_{j=1}^{n-1} \binom{8}{j}\binom{a+7-4k}{n-1-j}, \ for\ w' \in B_2 \quad \text{....(3.15)}$$

If $w' \in B_3$, then there are p-8k+3 vertices each of distance more than k from w′. Thus

$$C_n(w', P_m^{*c}, k) = \sum_{j-1}^{n-1} \binom{8}{j}\binom{a+3-4k}{n-1-j}, \ for\ w' \in B_3 \quad \text{....(3.16)}$$

Therefore, from (3.14),(3.15) and (3.16) we obtain the number of pairs (w', S), $w' \in B$, such that $d_n(w', S) = k$, which is

$$(2m+10-4k)\left[\binom{a+23-4k}{n-1} - \binom{a+15-4k}{n-1}\right] + (m+3-2k)\left[\binom{a+15-4k}{n-1} - \binom{a+7-4k}{n-1}\right] + (m+2-2k)\left[\binom{a+11-4k}{n-1} - \binom{a+3-4k}{n-1}\right]. \quad \text{...(3.17)}$$

Hence from (3.13) and (3.17)we obtain $C_n(P_m^{*c}, k)$, $4 \le k \le \left\lceil \frac{m}{2} \right\rceil$,as given in (3.8). ■

Proposition 3.5: For $3 \le n \le p = 4m+1$, $m \ge 7$,

$$C_n(P_m^{*c}, \left\lceil \frac{m}{2} \right\rceil + 1) = 8\binom{4m-4\lceil m/2\rceil + 4}{n-1} - 4\left[\binom{4m-8\lceil m/2\rceil + 4}{n-1} + \binom{4m-8\lceil m/2\rceil + 12}{n-1}\right] + (2m+6-4\lceil m/2\rceil)\left[\binom{4m-8\lceil m/2\rceil + 16}{n-1} - \binom{4m-8\lceil m/2\rceil + 8}{n-1}\right] + \bar{R}, \quad \text{...(3.18)}$$

where $\bar{R} = \begin{cases} 2\binom{4}{n-1} + \binom{8}{n-1}, & if\ m\ is\ even \\ zero, & if\ m\ is\ odd \end{cases}$

Proof: As in the proof of Proposition 3.4, we partition the vertices of P_m^{*c} in to A and B . If $w \in A_1 \cup A_2 \cup A_2' \cup A_3 \cup A_3'$, the number of pairs (w, S) such that $d_n(w, S) = \left\lceil \frac{m}{2} \right\rceil + 1$ are obtained from (3.9) , (3.10) , and (3.11). If m is even then there are only v_{m-1} and v_{m+1} such that $d_n(v_{m-1}, S) = d_n(v_{m+1}, S) = \frac{m}{2} + 1$, and

$$C_n(v_{m-1}, P_m^{*c}, \frac{m}{2}+1) = d_n(v_{m+1}, P_m^{*c}, \frac{m}{2}+1) = \binom{4}{n-1}.$$

If m is odd there is no vertex w in A_4 such that

$$d_n(w, S) = \left\lceil \frac{m}{2} \right\rceil + 1 = \frac{m+3}{2}.$$

If $w \in B_1$, the number of pairs (w, S) such that $d_n(w, S) = \left\lceil \frac{m}{2} \right\rceil + 1$ is obtained from (3.14) by putting $k = \left\lceil \frac{m}{2} \right\rceil + 1$. For m even only, there is one vertex of B_2, namely v_m, such that $d_n(v_m, S) = \frac{m}{2} + 1$,

and $C_n(v_m, P_m^{*c}, \frac{m}{2} + 1) = \binom{8}{n-1}$.

Moreover, there is no vertex w of $B_2 \cup B_3 - \{v_m\}$ such that

$d_n(w, S) = \left\lceil \frac{m}{2} \right\rceil + 1$.

Simplifying the result mentioned before we obtain (3.18). ■

Proposition 3.6: For $3 \le n \le p = 4m+1$, $m \ge 7$,

$$C_n(P_m^{*c}, \left\lceil \frac{m}{2} \right\rceil + 2) = 8\binom{4m - 4\lceil m/2 \rceil}{n-1} - 4\binom{4m - 8\lceil m/2 \rceil - 4}{n-1} + \begin{cases} 2\binom{8}{n-1} & , \text{ if } m \text{ is even} \\ \text{zero} & , \text{ if } m \text{ is odd} \end{cases}. \quad \ldots(3.19)$$

Proof: When $k = \left\lceil \frac{m}{2} \right\rceil + 2$, and $w \in A_1 \cup A_2 \cup A_2'$ we have the same results obtained Case (1) of the proof of Proposition 3.4.

Now we consider $D = \{v_i : i = 1, 2, \ldots, 2(m - \left\lceil \frac{m}{2} \right\rceil) - 2\}$, and

$D' = \{v_i : i = 2m-1, 2m-2, \ldots, 2\left\lceil \frac{m}{2} \right\rceil + 2\}$.

Then there are $a + 3 - 4\left\lceil \frac{i}{2} \right\rceil$ vertices each of distance more than $\left\lceil \frac{m}{2} \right\rceil + 2$ from $w \in D$, where $a = p - 4\left\lceil \frac{m}{2} \right\rceil - 8$. Thus

$$C_n(w, P_m^{*c}, \left\lceil \frac{m}{2} \right\rceil + 2) = \sum_{j-1}^{n-1} \binom{4}{j} \binom{a + 3 - 4\lceil i/2 \rceil}{n-1-j},$$

for $w \in D$.

Hence

$$\sum_{i=1}^{2(\lfloor m/2 \rfloor - 1)} C_n(v_i, P_m^{*c}, \left\lceil \frac{m}{2} \right\rceil + 2)$$

$$= \sum_{i=1}^{2(\lfloor m/2 \rfloor - 1)} \sum_{j-1}^{n-1} \binom{4}{j} \binom{a + 3 - 4\lceil i/2 \rceil}{n-j-1}$$

$$= 2\sum_{i=1}^{\lfloor m/2 \rfloor - 1} \sum_{j-1}^{n-1} \binom{4}{j} \binom{a + 3 - 4i}{n-1-j}$$

$$= 2\sum_{i=1}^{\lfloor m/2 \rfloor - 1} \left[\binom{a + 7 - 4i}{n-1-j} - \binom{a + 3 - 4i}{n-1-j} \right]$$

$$= 2\left[\binom{a+3}{n-1} - \binom{0}{n-1} \right] = 2\binom{a+3}{n-1}.$$

Since

$$C_n(v_i, P_m^{*c}, \left\lceil \frac{m}{2} \right\rceil + 2) = C_n(v_{2m-i}, P_m^{*c}, \left\lceil \frac{m}{2} \right\rceil + 2),$$

$i = 1, 2, \ldots, 2(\left\lfloor \frac{m}{2} \right\rfloor - 1)$, and therefore the number of pairs (w, S), $w \in D \cup D'$, such that:

$d_n(w, S) = \left\lceil \frac{m}{2} \right\rceil + 2$ is given by $4\binom{\alpha + 3}{n-1}$.

Finally from Fig.3.2, we notice that

$$C_n(u_{\frac{m}{2}+1}^{(r)}, P_m^{*c}, \left\lceil \frac{m}{2} \right\rceil + 2) = \binom{8}{n-1}, r = 0, 1, \ m \text{ is even}$$

This completes the proof. ■

Proposition 3.7: For $3 \le n \le p = 4m+1$, $\left\lceil \frac{m}{2} \right\rceil + 3 \le k \le d_n$, $m \ge 7$, then the coefficient of x^k in the n-Hosoya polynomial of P_m^{*c} is

$$C_n(P_m^{*c}, k) = 8\binom{a+7}{n-1}, \quad \ldots(3.20)$$

where $a = p - 4k$.

Proof: We assume that $F = F_1 \cup F_2 \cup F_2' \cup F_3 \cup F_3'$, where

$F_1 = \{u_i^{(r)} : r = 0, 1; i = 1, 2, m+1, m\}$,

$F_2 = \{u_i^{(r)} : r = 0\,,1\,; i = 3,\,4,\,\ldots\,,m-k+3\}$,

$F_2' = \{u_i^{(r)} : r = 0\,,1\,; i = m-1, m-2,\,\ldots\,,k-1\}$,

$F_3 = \{v_i :\ i = 1,\,2,\,\ldots\,,2(m-k)+2\}$,

$F_3' = \{v_i :\ i = 2m-1, 2m-2,\,\ldots\,,2k-2\}$.

For each $w'' \in F$, there are four vertices each of distance k from w'', $\left\lceil \frac{m}{2} \right\rceil + 3 \le k \le d_n - 1$.

If $w'' \in F_1$, then there are p-4k+3 vertices each of distance more than k from w''. Thus

$$C_n(w'', P_m^{*c}, k) = \sum_{j=1}^{n-1} \binom{4}{j} \binom{a+3}{n-1-j}\ ,\ for\ w'' \in F_1\,. \quad \ldots(3.21)$$

If $w'' \in F_2$, then there are p-4k+11-4i vertices each of distance more than k from w''. Thus

$$C_n(w'', P_m^{*c}, k) = \sum_{j=1}^{n-1} \binom{4}{j} \binom{a+11-4i}{n-1-j},\ for\ w'' \in F_2 \quad \ldots.(3.22)$$

If $w'' \in F_3$, then there are $p - 4k + 3 - 4\left\lceil \frac{i}{2} \right\rceil$ vertices each of distance more than k from w''. Thus

$$C_n(w'', P_m^{*c}, k) = \sum_{j=1}^{n-1} \binom{4}{j} \binom{a+3-4\lceil i/2 \rceil}{n-1-j},\ for\ w'' \in F_3 \quad \ldots.(3.23)$$

Since $C_n(u_i^{(r)}, P_m^{*c}, k) = C_n(u_{m-i+2}^{(r)}, P_m^{*c}, k)$, $i = 3,4,\ldots\ ,m-k+3,\ r = 0,1$, and

$C_n(v_i, P_m^{*c}, k) = C_n(v_{2m-i}, P_m^{*c}, k)$, $i = 1,2,\ \ldots\ ,2(m-k)+2$.

Therefore the number of pairs (w,S), $w \in F$, such that $d_n(w,S) = k$, $\left\lceil \frac{m}{2} \right\rceil + 3 \le k \le d_n - 1$, is given by:

$$8\left[\binom{a+7}{n-1} - \binom{a+3}{n-1}\right] + 2\sum_{i=3}^{m-k+3}\left[\binom{a+15-4i}{n-1} - \binom{a+11-4i}{n-1}\right]$$

$$+ 2\sum_{i=1}^{2(m-k)+2}\left[\binom{a+7-4\lceil i/2 \rceil}{n-1-j} - \binom{a+3-4\lceil i/2 \rceil}{n-1-j}\right]$$

$$= 8\left[\binom{a+7}{n-1} - \binom{a+3}{n-1}\right] + 4\left[\binom{a+3}{n-1} - \binom{0}{n-1}\right]$$

$$+ 4\left[\binom{a+3}{n-1} - \binom{0}{n-1}\right] = 8\binom{a+7}{n-1}.$$

Finally , we note from Fig. 3.2, that $C_n(P_m^{*c}, d_n) = 8\binom{4}{n-1}$.

This completes the proof. ■

Remark 2:

1. $C_n(P_m^{*c}, m+1) = 8\binom{4}{n-1}\ ,\ m \ge 3$.
2. $C_n(P_4^{*c}, 4) = 10\binom{8}{n-1} - 4\binom{4}{n-1}$.
3. $C_n(P_5^{*c}, 4) = 12\binom{12}{n-1} - 4\binom{8}{n-1} - 4\binom{4}{n-1}$
4. $C_n(P_6^{*c}, 4) = 14\binom{16}{n-1} - 4\binom{12}{n-1}$
   $-5\binom{8}{n-1} - 2\binom{4}{n-1}$.
5. $C_n(P_m^{*c}, m) = 8\binom{8}{n-1}\ ,\ m \ge 5$.
6. $C_n(P_6^{*c}, 5) = 8\binom{12}{n-1} + 2\binom{8}{n-1} - 4\binom{4}{n-1}$

From Propositions (3.2)- (3.7), and Remarks 1 and 2 , we obtain the next theorem .

Theorem 3.8: For P_m^{*c} of order $p=4m+1$, $m \ge 2$, and for $3 \le n \le p$, the n-Hosoya polynomial of P_m^{*c} is

$$H_n(P_m^{*c};x)=\sum_{k=0}^{d_n} C_n(P_m^{*c},k)x^k ,$$

where δ_n is the n-diameter of P_m^{*c} given in Proposition 1 , and

$$C_n(P_m^{*c},0)=p\binom{p-1}{n-2},$$

$$C_n(P_m^{*c},1)=p\binom{p-1}{n-1}-(2m-1)\binom{p-5}{n-1}-2(m+1)\binom{p-2}{n-1}$$

, and $C_n(P_m^{*c},2)$, $C_n(P_m^{*c},3)$, are respectively (3.1) , (3.2), and $C_n(P_m^{*c},k)$, $4 \le k \le d_n$ is given in Propositions (3.4) - (3.7). ■

Corollary 3.9: For P_m^{*c} of order 4m+1, we have

$$H(P_m^{*c};x)=(4m+1)+(5m-1)x+3(3m-1)x^2 +12(m-1)x^3+16\sum_{k=4}^{m+1}(m+2-k)x^k .$$

Proof: For n=2, we have $d_2(u,\{v\})=d_2(v,\{u\})=d(u,v)$.

Thus $H(P_m^{*c};x)$ is obtain from Propositions (3.2)-(3.7) and Remarks 1 and 2 by putting n=2 and dividing by 2. From Corollary 3.9 , we note that the sequence $(C(P_m^{*c},k))$, $k=0,1,\ \dots\ ,d$ is strong-unimodal, $m>5$, since there is index h=4, such that

$$C(P_m^{*c},0)<C(P_m^{*c},1)<C(P_m^{*c},2)<C(P_m^{*c},3)> C(P_m^{*c},4)>C(P_m^{*c},5)>\dots>C(P_m^{*c},m+1).$$

Hence , the sequence $(C(P_m^{*c},k))$, $k=0,1,\ \dots\ ,m+1$, $m>5$, is neither palindromic nor semi-palindromic.

Corollary 3.10: The Wiener index of thorn saw graph P_m^{*c} of order 4m+1, is

$$W(P_m^{*c})=16\binom{m+3}{3}-(37m+11).$$ ■

Proof: By taking the derivative of the Hosoya polynomial of P_m^{*c} of order 4m+1 determined in the Corollary 3.9, with respect to x and then putting x=1, we have

$$W(P_m^{*c})=16\binom{m+3}{3}-(37m+11).$$ ■

REFRENCE

[1]. Ali , A.A. , and Ali , A.M. ; (2006). "Wiener Polynomials of the Generalized Distance for some special Graphs" , Raf. J. Comp. Sc. And Maths. Vol.3 , No.2 , pp.103-120.

[2]. Ali , A.A. , and Ahmed , H.G. ; (2007). "n-Wiener Polynomials of the m-Cube , Wagner Graphs and Thorn Stars" , J. Dohuk University , Vol.10 , No.2 ,

[3]. Buckley , F. and Harary , F.; (1990). **Distance in Graphs**. Addison -Wesley, Redwood , California .

[4]. Chartrand G. and Lesniak , L.; (1986). **Graphs and Digraphs**, 2nd. , Wadsworth and Brooks / Cole, California .

[5]. Saeed , W.A.M. ; (1999) . " Wiener Polynomials of Graphs " , Ph.D. thesis , Mosul University , Mosul.