Unitarily Invariant Norm Inequalities for Sum of Operators

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ABSTRACT

For Hilbert space operators A and B, Bhatia and Kittaneh proved that $|||A^*B+B^*A||| \leq |||A^*A+B^*B|||$ for every unitarily invariant norm. A generalization of this inequality to n-tuple of operators is obtained from a general inequality for sum of Hilbert space operators. Among other inequalities, it is shown that if A_i, B_i and X_i $(i=1,2,....,n)$ are operators in $B(H)$ such that X_i is positive, then

$$|||\sum_{i=1}^{n}\left(A_i^*X_iB_i+B_i^*X_iA_i\right)||| \leq |||\sum_{i=1}^{n}\left(A_i^*X_iA_i+B_i^*X_iB_i\right)|||.$$

Other related norm inequalities for sum of operators and applications to the arithmetic-geometric mean inequality are also discussed.

Keywords: Unitarily invariant norm; Norm inequality; n-tuple of operators; Hölder’s inequality; Arithmetic-geometric mean inequality.

1. Introduction

Let H be a complex separable Hilbert space, and let $B(H)$ denote the C^*-algebra of all bounded linear operators on H. A function $|||\,.\,|||: B(H)\rightarrow \mathbb{R}$ is called a unitarily invariant norm if it is a norm satisfying the invariance property $|||UAV|||=|||A|||$ for all unitary operators U and V in $B(H)$. Except the operator norm, which is defined on all of $B(H)$, each unitarily invariant norm is defined on a norm ideal associated with it and contained in $B(H)$. When we use the symbol $|||A|||$ it is implicit that the operator A belongs to the class of operators on which this norm is defined. For the theory of unitarily invariant norms we refer to [2], [5], and [10].

The absolute value of $A\in B(H)$, denoted by $|A|$, is the positive operator $\left(A^*A\right)^{1/2}$. If A is a compact operator on a Hilbert space H, then the singular values of A are defined to be the eigenvalues of the positive operator $|A|$ enumerated as $s_1(A)\geq s_2(A)\geq ....$ Typical examples of the unitarily invariant norm are the usual operator norm $\|A\|=s_1(A)$, the trace norm $\|A\|_1=\sum_{j=1}^{\infty}s_j(A)$, and the Frobennius norm $\|A\|_2=\left(\sum_{j=1}^{\infty}s_j^2(A)\right)^{1/2}$. These norms are special examples of the more general class of the Schatten p-norm, defined by $\|A\|_p = \left(\sum_{j=1}^{\infty} s_j^p(A)\right)^{1/p}$. For basic properties of the Schatten p-norm, we refer to [8] and [9].

The unitarily invariant norm inequalities for sum of Hilbert space operators are important and applicable in analysis, and they have been generalized in various directions. It has been shown by Bhatia and Kittaneh in [4] that if A and B are operators in $B(H)$, then

$$|||A^*B + B^*A||| \leq |||A^*A + B^*B|||, \quad (1)$$

for every unitarily invariant norm. This inequality has been obtained as an application of Cauchy-Schwarz inequality, and can be considered as non-commutative version of the familiar arithmetic-geometric mean inequality for real numbers. The inequality in (1) has been refined in [7] by: If A and B are operators in $B(H)$, then

$$2\left|\left|\left|\begin{bmatrix} A^*B + B^*A & 0 \\ 0 & 0 \end{bmatrix}\right|\right|\right| \leq \left|\left|\left|\begin{bmatrix} (A+B)^*(A+B) & 0 \\ 0 & (A-B)^*(A-B) \end{bmatrix}\right|\right|\right|, \quad (2)$$
$$\leq 2\left|\left|\left|\begin{bmatrix} A^*A + B^*B & 0 \\ 0 & 0 \end{bmatrix}\right|\right|\right|$$

for every unitarily invariant norm.

Stronger versions of inequality (1) have been proved in [7] assert that: If A, B and X are operators in $B(H)$ such that X is a normal operator, then

$$|||\,|A^*XB + B^*XA|^r\,||| \leq |||\left(A^*|X|A + B^*|X|B\right)^r|||, \quad (3)$$

for every $r > 0$ and for every unitarily invariant norm. In particular, if X is positive,then

$$|||A^*XB + B^*XA||| \leq |||A^*XA + B^*XB|||. \quad (4)$$

For basic properties of these inequalities and related inequalities for sum of operators, we refer to [3], [6] and references therein.

In this paper, we present norm inequalities for sum of Hilbert space operators that generalizes (1), (3), and (4). These inequalities seem natural enough to be useful. Our analysis is based on Hölder's inequality for operators and other classical inequalities. Relations between these inequalities and the different forms of the arithmetic-geometric mean inequality for operators are also obtained.

2. Norm Inequalities for Sum of Operators

In this section, we establish a general unitarily invariant norm inequality for sum of Hilbert space operators, from which well-known and new norm inequalities for operators follow as special cases. Related inequalities are also derived.

To prove our generalized inequality we need the following lemma, which is a generalized form of Hölder's inequality, and was proved by the author in [1].

Lemma 1. *If A, B, and X are operators in $B(H)$, then*

$$|||\,|A^*XB + B^*XA|^r\,||| \leq |||\left(A^*|X^*|A + B^*|X^*|B\right)^{\frac{pr}{2}}|||^{\frac{1}{p}}\,|||\left(A^*|X|A + B^*|X|B\right)^{\frac{qr}{2}}|||^{\frac{1}{q}} \quad (5)$$

for every positive real numbers r, p and q with $p^{-1}+q^{-1}=1$, and for every unitarily invariant norm.

Now we are in a position to establish a general inequality, from which we obtain our stronger version of inequality (3).

Theorem 1. *Let A_i, B_i and X_i be operators in $B(H)$ $(i = 1, 2, ..., n)$. If p, q and r are positive real numbers such that $p^{-1}+q^{-1}=1$, then*

$$|||\,\left|\sum_{i=1}^{n}\left(A_i^*X_iB_i + B_i^*X_iA_i\right)\right|^r\,||| \leq \left(|||\left(\sum_{i=1}^{n}\left(A_i^*|X_i^*|A_i + B_i^*|X_i^*|B_i\right)\right)^{\frac{pr}{2}}|||^{\frac{1}{p}}\right) \cdot \left(|||\left(\sum_{i=1}^{n}\left(A_i^*|X_i|A_i + B_i^*|X_i|B_i\right)\right)^{\frac{qr}{2}}|||^{\frac{1}{q}}\right), \quad (6)$$

for every unitarily invariant norm.

Proof. *Let*

$$A=\begin{bmatrix} A_1 & 0 & \cdots & 0 \\ A_1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_n & 0 & \cdots & 0 \end{bmatrix},\quad B=\begin{bmatrix} B_1 & 0 & \cdots & 0 \\ B_1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ B_n & 0 & \cdots & 0 \end{bmatrix},$$

$$X=\begin{bmatrix} X_1 & 0 & \cdots & 0 \\ 0 & X_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \vdots & X_n \end{bmatrix}.\ \textit{Then}$$

$$A^*XB=\begin{bmatrix} \sum_{i=1}^{n} A_i^*X_iB_i & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix},$$

$$B^*XA=\begin{bmatrix} \sum_{i=1}^{n} B_i^*X_iA_i & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix},$$

$$A^*|X|A=\begin{bmatrix} \sum_{i=1}^{n} A_i^*|X_i|A_i & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix},$$

$$A^*|X^*|A=\begin{bmatrix} \sum_{i=1}^{n} A_i^*|X_i^*|A_i & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix},$$

$$B^*|X|B=\begin{bmatrix} \sum_{i=1}^{n} B_i^*|X_i|B_i & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix},$$

$$B^*|X^*|B=\begin{bmatrix} \sum_{i=1}^{n} B_i^*|X_i^*|B_i & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Now inequality (6) *follows from inequality* (5).

Note that many consequences are established by choosing certain values of *p*, *q*, *n*, and special cases for *X* in inequality (6). Some of these consequences are demonstrated below, other inequalities of this genre are left to the interested reader.

Choosing $p=q=2$ in inequality (6) gives the following inequality.

Corollary 1. *Let* A_i, B_i *and* X_i $(i=1,2,....,n)$ *be operators in* $B(H)$, *and let* $r>0$. *Then*

$$|||\,|\sum_{i=1}^{n}\left(A_i^*X_iB_i+B_i^*X_iA_i\right)|^r\,|||^2 \le \left(|||(\sum_{i=1}^{n}\left(A_i^*|X_i^*|A_i+B_i^*|X_i^*|B_i\right))^r\,|||\right).\left(|||(\sum_{i=1}^{n}\left(A_i^*|X_i|A_i+B_i^*|X_i|B_i\right))^r\,|||\right),\ (7)$$

for every unitarily invariant norm.

Remark 1. *Inequality* (7) *gives a generalized form of inequality* (3), *it has been proved in* [7] *that: If* A, B, *and* X *are operators in* $B(H)$, *then*

$$|||\,|A^*XB+B^*XA|^r\,|||^2 \le |||(A^*|X^*|A+B^*|X^*|B)^r\,|||.|||(A^*|X|A+B^*|X|B)^r\,|||\qquad (8)$$

for every $r>0$ *and for every unitarily invariant norm, which can be obtained directly by letting* $n=1$ *in inequality* (7).

When X_i is a normal operator in inequality (7), we have the following inequality.

Corollary 2. *Let* A_i, B_i *and* X_i *be operators in* $B(H)$ *such that* X_i *is normal* $(i=1,2,....,n)$, *and let* $r>0$. *Then*

$$|||\,|\sum_{i=1}^{n}\left(A_i^*X_iB_i+B_i^*X_iA_i\right)|^r\,||| \le |||(\sum_{i=1}^{n}\left(A_i^*|X_i|A_i+B_i^*|X_i|B_i\right))^r\,|||,\ (9)$$

for every unitarily invariant norm.

Remark 2. *Inequality* (9) *gives a generalized form of inequality* (4), *and gives inequality* (3) *by letting* $n=1$ *in inequality* (9). *A stronger version of inequality* (1) *can be obtained by letting*

$X = I$ *in inequality* (9). *In fact if* A_i, B_i $(i = 1, 2, \ldots, n)$ *are operators in* $B(H)$, *and* $r > 0$. *Then*

$$|||\,|\sum_{i=1}^{n}\left(A_i^*B_i + B_i^*A_i\right)|^r\,||| \le |||(\sum_{i=1}^{n}\left(A_i^*A_i + B_i^*B_i\right))^r\,||| , \quad (10)$$

for every unitarily invariant norm.

By choosing $r = 1$ and letting X_i positive operator in inequality (9), we get the following inequality that generalizes (4) to *n*-tuple of operators.

Corollary 3. *Let* A_i, B_i *and* X_i $(i = 1, 2, \ldots, n)$ *be operators in* $B(H)$ *such that* X_i *is positive. Then*

$$|||\sum_{i=1}^{n}\left(A_i^*X_iB_i + B_i^*X_iA_i\right)||| \le |||\sum_{i=1}^{n}\left(A_i^*X_iA_i + B_i^*X_iB_i\right)||| , \quad (11)$$

for every unitarily invariant norm.

Remark 3. *Inequality* (11) *gives inequality* (4) *in the case* $n = 1$, *while the choice* $X_i = I$ *in inequality* (11) *gives the following natural generalization of inequality* (1), *which asserts that:*

$$|||\sum_{i=1}^{n}\left(A_i^*B_i + B_i^*A_i\right)||| \le |||\sum_{i=1}^{n}\left(A_i^*A_i + B_i^*B_i\right)||| , \quad (12)$$

for every unitarily invariant norm.

Finally, we end this paper by the following remark.

Remark 4. *In view of the inequalities* (11) *and* (12), *it is reasonable to conjecture that if A, B and X are operators in* $B(H)$, *then*

$$|||A^*XB + B^*XA||| \le |||AXA^* + BXB^*|||$$

or

$$|||A^*XB + B^*XA||| \le |||A^*AX + XB^*B|||$$

for every positive operator X, However, this conjecture is refuted by the example: $A = X = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, *and* $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ *for which*

$$\|A^*XB + B^*XA\|_1 > \|AXA^* + BXB^*\|_1 \text{ and}$$

$$\|A^*XB + B^*XA\|_1 > \|A^*AX + XB^*B\|_1 .$$

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