A REFINED SOLUTION TO THE PROBLEM OF TORSION OF A VISCOELASTIC ANISOTROPIC REINFORCED LAYER

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ABSTRACT:

Torsion of anisotropic viscoelastic reinforced layer represents both crucial and challenge process in numerous of composite materials within distinct real-world engineering industry. This paper proposes a novel innovative scheme for formal solution of the stress and displacement, which is occurring in an infinite anisotropic reinforced viscoelastic layer, when it is twisted by means of turning a rigid cylindrical shaft attached to it. In this research, we modify the problem by adding some important terms within the proposed solution and assuming that the layer is made of composite viscoelastic material. The entire technique for solving quasi-static viscoelastic problems in a reinforced composite material with the method of effective module for this model is proposed. The motivation behind such idea is to find the distributions of stresses and displacement in our model. The behavior of the layer is governed by the equilibrium equation are solved by means of Hankel transforms. The dynamic mixed boundary value problem can lead to dual integral equations as a first step. The solution of purely elastic layer is obtained, and then the problem of model with anisotropic reinforced viscoelastic layer is solved using the correspondence principle and Ilyushin's approximation method. A numerical example of the fields of stress and displacement are illustrated with diagrams which compare the cases of anisotropic materials with those of an isotropic material.

Keywords

Torsion, Shafts, Viscoelasticity, Anisotropic material, Elastic layer, Dual integral equation.

Introduction

In recent years, composite materials have gained considerable attention in many engineering applications. Composite materials are considered as a structural material for future high-speed spacecraft and power generation industries. The composite materials are microscopically inhomogeneous, in which the mechanical properties vary from one point to the other or one surface to another.

The general stability of drive shafts under torsion has been studied by many researchers. Greenhill [6] for the first time in 1883 presented a solution for torsion stability of long solid shafts. This method of solution can be extended for calculating of the first torsion buckling mode of a hollow shaft. Tennyson [12] using a theoretical method studied the classical linear elastic buckling of non-isotropic composite cylinders, "perfect" and "imperfect", under different loading conditions. He compared his results with experiments. Bauchau and Krafchack [1] in 1988 measured the torsional buckling load of some composite drive shafts made of carbon / epoxy. They predicated the torsional buckling load using shell theory and by considering the effects of elastic coupling and transverse shear deformation. Chen and Peng [6] in 1998, using a finite element method, studied the stability of composite shafts under rotation and axial comparison load. They predicated the critical axial load of a thin-wall composite shaft under rotation. The torsional problem of an anisotropic layer with fixed base on a rigid foundation when it is twisted by means of turning an attached rigid cylindrical shaft studied by Tang [10]. In this problem we modify the problem by adding some terms in the solution which are cancel in the solution [10], and then we suppose that the layer is made of composite viscoelastic material. Methods of solving quasi-static viscoelastic problems in a reinforced composite material have been developed by Allam and Pobedria [9]. Allam and Zenkour [8] have used the small parameter method as well as the method of effective module for the bending response of a reinforced viscoelastic arched bridge model. The main objective being to find the distributions of stresses and displacement in our model.

Formulation of the Problem

Consider the anisotropic layer whose material has three mutually perpendicular directions of elasticity symmetry parallel to the axis of coordinates. For such material the equations of the generalized Hooke's law is written in the following way [11]:

$$\varepsilon_{r} = \frac{\partial u_{r}}{\partial r} = s_{11}\sigma_{r} + s_{12}\sigma_{\theta} + s_{13}\sigma_{z} \qquad \varepsilon_{\theta z} = \frac{\partial u_{\theta}}{\partial z} + \frac{1}{r}\frac{\partial u_{z}}{\partial \theta} = s_{44}\sigma_{\theta z}$$

$$\varepsilon_{\theta} = \frac{1}{r}\frac{\partial u_{\theta}}{\partial \theta} + \frac{u_{r}}{r} = s_{12}\sigma_{r} + s_{22}\sigma_{\theta} + s_{23}\sigma_{z} \qquad \varepsilon_{zr} = \frac{\partial u_{z}}{\partial r} + \frac{\partial u_{r}}{\partial z} = s_{55}\sigma_{zr} \qquad (1)$$

$$\varepsilon_{z} = \frac{\partial u_{z}}{\partial z} = s_{13}\sigma_{r} + s_{23}\sigma_{\theta} + s_{33}\sigma_{z} \qquad \varepsilon_{r\theta} = \frac{1}{r}\frac{\partial u_{r}}{\partial \theta} + \frac{\partial u_{\theta}}{\partial r} - \frac{u_{\theta}}{r} = s_{66}\sigma_{r\theta}$$

where s_{ij} are elastic constant's of the layer, u_r , u_{θ} and u_z are the displacements with references to (r, θ, z) coordinates respectively, ε_r , ε_{θ} , ε_z , $\varepsilon_{\theta z}$, $\varepsilon_{r\theta}$ and ε_{rz} are the strain components in cylindrical coordinate, σ_r , σ_{θ} , σ_z , $\sigma_{\theta z}$, $\sigma_{r\theta}$ and σ_{rz} are the stress components in cylindrical coordinate.

Let us take (see Fig.1) the axis of the cylindrical shaft as the Z -axis of a cylindrical coordinate system (r, θ, z) the origin being at the centre of the cross section of the shaft which is attached to the layer, a is the radius of the cylindrical shaft and h is the thickness of the layer. The polar r-axis is directed arbitrarily. It is assumed that one of the boundary surface (z = h) is fixed and

the other surfaces of the layer (z = 0) is subjected to a torsion Ω over the area $0 \le r \le a$ and let $a \le h$.

It is assumed that the cross sections of the layer are not rotated and the displacements in the radial and axial directions are absent, that is:

$$u_r = 0$$
 , $u_\theta = u_\theta(r, z)$, $u_z = 0$

then from (1) all the components of stress vanish identically except $\sigma_{\theta z}$ and $\sigma_{r\theta}$ which are given by the relations

$$\sigma_{\theta z} = \frac{1}{s_{44}} \frac{\partial u_{\theta}}{\partial z} \quad \& \quad \sigma_{r\theta} = \frac{1}{s_{66}} \left(\frac{\partial u_{\theta}}{\partial r} - \frac{u_{\theta}}{r} \right)$$
(2)

the remaining two components of stress $\sigma_{\theta z}$ and $\sigma_{r\theta}$ depend only on r and z, and by using the equations of equilibrium [11]:

$$\frac{\partial \sigma_r}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{\partial \sigma_{zr}}{\partial z} + \frac{\sigma_r - \sigma_{\theta}}{r} = 0$$
$$\frac{\partial \sigma_{r\theta}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{\partial \sigma_{\thetaz}}{\partial z} + 2 \frac{\sigma_{r\theta}}{r} = 0$$
$$\frac{\partial \sigma_{zr}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{\thetaz}}{\partial \theta} + \frac{\partial \sigma_z}{\partial z} + \frac{\sigma_{zr}}{r} = 0$$

The first and third equations are directly vanished, and the second equation becomes:

$$\frac{\partial^2 u_{\theta}}{\partial r^2} + \frac{1}{r} \frac{\partial u_{\theta}}{\partial r} - \frac{u_{\theta}}{r^2} + k \frac{\partial^2 u_{\theta}}{\partial z^2} = 0, \ k = \frac{s_{66}}{s_{44}}$$
(3)

the mixed boundary conditions are assumed to be:

$$u_{\theta}|_{r=0} = 0 \tag{4.a}$$

$$u_{\theta}\big|_{z=h} = 0 \tag{4.b}$$

$$u_{\theta}\Big|_{\substack{z=0\\r< a}} = \Omega r \tag{4.c}$$

$$\frac{\partial u_{\theta}}{\partial z}\Big|_{\substack{z=0\\r>a}} = s_{44}\sigma_{\theta z}\Big|_{\substack{z=0\\r>a}} = 0$$
(4.d)

where Ω is the angular displacement of the shaft. The solution of the equation (3) has the form:

$$u_{\theta}(r,z) = \int_{0}^{\infty} \left(B_{1} \cosh\left(\frac{p z}{\sqrt{k}}\right) + B_{2} \sinh\left(\frac{p z}{\sqrt{k}}\right) \right) J_{1}(pr) dp$$

where B_1, B_2 is a constant and J_1 is the first Bessel function. From boundary conditions (4.a) and (4.b) and assuming that at infinity $(r \rightarrow \infty)$ the displacement u_{θ} must remain finite then by the use of Hankel transform theorem, the general solution of the differential equation takes the form:

$$u_{\theta}(r,z) = \int_{0}^{\infty} A(p) \frac{\sinh\left[\frac{p}{\sqrt{k}}(h-z)\right]}{\sinh\left(\frac{ph}{\sqrt{k}}\right)} J_{1}(pr)dp$$
(5)

where A(p) is the unknown function determine from boundary conditions.

It follows from (2) that the non-vanishing components of stress are:

$$\sigma_{\theta z} = \frac{1}{s_{44}} \frac{\partial u_{\theta}}{\partial z} = -\frac{1}{s_{44}} \int_{0}^{\infty} p A(p) \frac{\cosh\left[\frac{p}{\sqrt{k}}(h-z)\right]}{\sinh\left(\frac{ph}{\sqrt{k}}\right)} J_{1}(pr) dp$$

$$\sigma_{r\theta} = \frac{1}{s_{66}} \left(\frac{\partial u_{\theta}}{\partial r} - \frac{u_{\theta}}{r}\right) = -\frac{1}{s_{66}} \int_{0}^{\infty} p A(p) \frac{\sinh\left[\frac{p}{\sqrt{k}}(h-z)\right]}{\sinh\left(\frac{ph}{\sqrt{k}}\right)} J_{2}(pr) dp$$
(6)
noting that [5]:
$$J_{2}(pr) = -\frac{1}{p} \frac{dJ_{1}(pr)}{dr} - \frac{J_{1}(pr)}{r}$$

where J_2 is the second order Bessel function.

Using the boundary conditions on z = 0 (4.c), (4.d) we have the equations:

$$\int_{0}^{\infty} A(p) J_{1}(pr) dp = \Omega r \qquad 0 \le r \le a$$

$$\int_{0}^{\infty} p A(p) \operatorname{coth}\left(\frac{ph}{\sqrt{k}}\right) J_{1}(pr) dp = 0 \qquad a < r < \infty$$
(7)

Putting $p A(p) \operatorname{coth}(pb) = f(p)$, where $b = \frac{h}{\sqrt{k}}$, then the equations (7) become the dual integration of the second equation equation of the second equation equation of the second equation e

integral equations:

$$\begin{bmatrix}
\int_{0}^{\infty} p^{-1} \tanh(pb) f(p) J_{1}(pr) dp = \Omega r & 0 \le r \le a \\
\int_{0}^{\infty} f(p) J_{1}(pr) dp = 0 & a < r < \infty
\end{bmatrix}.$$
(8)

Using Tranter's method [3], we have

$$f(p) = \frac{\Omega(2p)^{1-\alpha}}{\Gamma(1+\alpha)} \sum_{n=0}^{\infty} (\delta_{n0} - c_n + c'_n - c''_n + \mathbf{L}) J_{2n+\alpha+1}(a p)$$
(9)

where α is a real and positive number. Γ is the gamma function, and δ_{n0} is the kronecker delta symbol and

$$c_n = L_{0,n}$$
 , $c'_n = \sum_{m=0}^{\infty} L_{m,n} c_m$, $c''_n = \sum_{m=0}^{\infty} L_{m,n} c'_m$,... (10)

for b > 1, we may choose $\alpha = \frac{1}{2}$ (Tranter [3]), then:

$$L_{m,n} = (4n+3) \int_{0}^{\infty} p^{-1} (\tanh(pb) - 1) J_{2m+\frac{3}{2}}(ap) J_{2n+\frac{3}{2}}(ap) dp$$
(11)

writing the hyperbolic function in this equation (11) as a series of exponentials, we have:

$$\frac{1}{(8n+6)}L_{m,n} = \sum_{q=1}^{\infty} (-1)^q \int_0^{\infty} p^{-1} e^{(-2qbp)} J_{2m+\frac{3}{2}}(ap) J_{2n+\frac{3}{2}}(ap) dp$$
(12)

The right hand side of this equation is a typical form of Weber-Schaftheitlin integral which can be expressed as a power series in $(2qh/\sqrt{k})^{-1}$ by using the formal given by Watson [4]. Doing this, interchange the order of summation.

$$\int_{0}^{\infty} e^{-2qbp} p^{-1} J_{2m+\frac{3}{2}}(ap) J_{2n+\frac{3}{2}}(ap) dp = \sum_{t=0}^{\infty} (-1)^{t} \frac{\Gamma(2n+2m+2t+3)\Gamma(2m+2n+2t+4)}{t!\Gamma(2m+t+\frac{5}{2})\Gamma(2n+t+\frac{5}{2})\Gamma(2m+2n+t+4)} \left(\frac{\beta}{4}\right)^{2m+2n+2t+3} q^{-2m-2n-2t-3}$$

where $\beta = \frac{a}{b} = \frac{a}{h}\sqrt{k}$, the equation (12) may written as:

$$\frac{1}{(8n+6)}L_{m,n} = \sum_{t=0}^{\infty} (-1)^{t+1} D_1 \left(\frac{\beta}{4}\right)^{2m+2n+2t+3} \left(1-2^{1-(2m+2n+2t+3)}\right) \zeta \left(2m+2n+2t+3\right)$$
(13)

where $D_1 = \frac{\Gamma(2n+2m+2t+3)\Gamma(2m+2n+2t+4)}{t!\Gamma(2m+t+\frac{5}{2})\Gamma(2n+t+\frac{5}{2})\Gamma(2m+2n+t+4)}$

and $\zeta(t)$ is Rieman's zeta function. Then $L_{m,n}$ may be determine to any degree of approximation according to the power of $\beta(\beta = 1)$, for the seventh power of β we have:

$$L_{0,0} = \frac{-1}{\pi} \left(\frac{1}{4} \beta^{3} \zeta(3) - \frac{3}{16} \beta^{5} \zeta(5) + \frac{81}{640} \beta^{7} \zeta(7) \right)$$

$$L_{0,1} = \frac{-1}{\pi} \left(\frac{1}{16} \beta^{5} \zeta(5) - \frac{49}{640} \beta^{7} \zeta(7) \right)$$

$$L_{1,0} = \frac{-1}{\pi} \left(\frac{3}{32} \beta^{5} \zeta(5) - \frac{189}{1280} \beta^{7} \zeta(7) \right)$$

$$L_{1,1} = \frac{-1}{\pi} \left(\frac{9}{640} \beta^{7} \zeta(7) \right)$$

$$L_{0,2} = \frac{-1}{\pi} \left(\frac{1}{128} \beta^{7} \zeta(7) \right)$$

$$L_{2,0} = \frac{-1}{\pi} \left(\frac{3}{1408} \beta^{7} \zeta(7) \right)$$
(14)

All the other L's are negligible since the power of β greater than 7. Using the results (14) in (10), we find c's existing they are:

$$c_{0} = -\frac{1}{4\pi} \left[\beta^{3} \zeta(3) - \frac{3}{4} \beta^{5} \zeta(5) + \frac{81}{160} \beta^{7} \zeta(7) \right]$$

$$c_{1} = -\frac{1}{16\pi} \left[\beta^{5} \zeta(5) - \frac{49}{40} \beta^{7} \zeta(7) \right]$$

$$c_{2} = -\frac{1}{128\pi} \beta^{7} \zeta(7)$$

$$c_{0}' = \frac{1}{16\pi^{2}} \beta^{6} \zeta(3) \zeta(3)$$

$$c_{1}' = c_{2}' = c_{0}'' = c_{1}'' = c_{2}'' = 0$$
(15)

Substitution of the above mentioned values in equation (10) yields:

$$f(p) = \Omega(2p)^{\frac{1}{2}} \sum_{n=0}^{\infty} \frac{1}{\Gamma(\frac{3}{2})} [\delta_{n0} - c_n + c'_n - c''_n + \mathbf{L}] J_{2n+\frac{3}{2}}(ap)$$

= $\Omega \sqrt{\frac{8p}{\pi}} \Big[[1 - c_0 + c'_0 - c''_0] J_{\frac{3}{2}}(ap) + [-c_1 + c'_1 - c''_1] J_{\frac{7}{2}}(ap) + [-c_2 + c'_2 - c''_2] J_{\frac{11}{2}}(ap) \Big]$
So by using equation (15)

So by using equation (15)

$$f(p) = \Omega_{\sqrt{\frac{8p}{\pi}}} \left[H_0 J_{\frac{3}{2}}(ap) + H_1 J_{\frac{7}{2}}(ap) + H_2 J_{\frac{11}{2}}(ap) \right]$$
(16)

Then the function A(p) can be found as:

$$A(p) = \Omega \sqrt{\frac{8}{\pi p}} \tanh(pb) \left[H_0 J_{\frac{3}{2}}(ap) + H_1 J_{\frac{7}{2}}(ap) + H_2 J_{\frac{11}{2}}(ap) \right]$$
(17)

where

$$H_{0} = 1 + \frac{1}{4\pi} \left(\beta^{3} \zeta(3) - \frac{3}{4} \beta^{5} \zeta(5) + \frac{81}{640} \beta^{7} \zeta(7) \right) + \frac{1}{16\pi^{2}} \beta^{6} \zeta(3) \zeta(3)$$
$$H_{1} = \frac{1}{16\pi} \left(\beta^{5} \zeta(5) - \frac{49}{40} \beta^{7} \zeta(7) \right); \quad H_{2} = \frac{-1}{128\pi} \beta^{7} \zeta(7)$$

The displacement and stresses can then be found in equation (6). When h is sufficiently large (i.e. for an elastic half-space) follows:

$$f(p) = 2\Omega\left(\frac{2p}{\pi}\right)^{\frac{1}{2}} J_{\frac{3}{2}}(ap)$$

using the a asymptotic expansions given by Watson [4],[5]. This equation may be written as:

$$f(p) = \frac{4\Omega}{\pi} \left(\frac{\sin(ap)}{\sqrt[3]{ap}} - \cos(ap) \right)$$

this is the expression for half-space given by Sneddon [7] in the isotropic case.

Numerical Solutions

The numerical solutions of stresses and displacement are directly evaluated from equation (5) and (6) with the result of f(p) and A(p) can be written by:

$$u_{\theta}(r,z) = \Omega \sqrt{\frac{8}{\pi}} \int_{0}^{\infty} p^{-\frac{1}{2}} \tanh(pb) \frac{\sinh\left[\frac{p}{\sqrt{k}}(h-z)\right]}{\sinh(pb)} D_{2} J_{1}(pr) dp$$

$$\sigma_{\theta z}(r,z) = \frac{-\Omega}{s_{44}} \sqrt{\frac{8}{\pi k}} \int_{0}^{\infty} p^{\frac{1}{2}} \tanh(pb) \frac{\cosh\left[\frac{p}{\sqrt{k}}(h-z)\right]}{\sinh(pb)} D_{2} J_{1}(pr) dp$$

$$\sigma_{r\theta}(r,z) = -\frac{\Omega}{s_{66}} \sqrt{\frac{8}{\pi}} \int_{0}^{\infty} p^{\frac{1}{2}} \tanh(pb) \frac{\sinh\left[\frac{p}{\sqrt{k}}(h-z)\right]}{\sinh(pb)} D_{2} J_{2}(pr) dp$$
(18)

Where $D_2 = H_0 J_{\frac{3}{2}}(a p) + H_1 J_{\frac{7}{2}}(a p) + H_2 J_{\frac{11}{2}}(a p)$

The numerical solutions of stresses and displacements are directly evaluated from the relation (4) in the appendix; the final form is as follows:

$$u_{\theta}(r,z) = \frac{2\Omega}{\sqrt{a}} \begin{cases} \frac{2}{3\pi} B_{1}(1) & r \leq a \\ \frac{r}{\sqrt{\pi}} B_{2}(1) & r \geq a \end{cases}$$

$$\sigma_{\theta z}(r,z) = -\frac{4\Omega}{s_{44}\sqrt{a \, k}} \begin{cases} \frac{1}{3\pi} B_{1}(2) & r \leq a \\ \frac{r}{\sqrt{\pi}} B_{2}(2) & r \geq a \end{cases}$$

$$\sigma_{r\theta}(r,z) = -\frac{\Omega}{s_{66}\sqrt{a}} \begin{cases} \frac{4}{3\pi} B_{1}(3) & r \leq a \\ \frac{r^{2}}{\sqrt{\pi}} B_{2}(3) & r \geq a \end{cases}$$

where $_{2}F_{1}(v,\mu;\gamma;\lambda)$ is the hypergeometric function,

$$c_1 = 2q b + \frac{z}{\sqrt{k}}$$
 and $c_2 = 2q b + 2b - \frac{z}{\sqrt{k}}$.

The expressions for $B_1(k)$, $B_2(k)$, k = 1, 2, 3 are given in the appendix

Viscoelastic Composite layer

Now, consider the solution for the case of a layer made of viscoelastic isotropic material (filler) reinforced by elastic isotropic fibers, this material is considered as structurally anisotropic material.By using Hook's law in anisotropic layer we have the relations:

$$S_{66} = \frac{1}{G_{12}}, \quad S_{44} = \frac{1}{G_{23}}, \quad E = 2G(1+\nu)$$
 (19)

where E_i is the Young's modulus G_i the shear modulus and v_i the Poisson's ratio where (i = 1, 2) is the elastic and viscoelastic material respectively.

Let the viscoelastic filler characterized by E, v or by the bulk modulus K and dimensionless kernel of relaxation $\omega(t)$ while the elastic reinforcement characterized by Young's modulus E₁

and Poisson ratio v_1 ; where

$$\upsilon_2 = \frac{1-\omega}{2+\omega} , \quad E_2 = \frac{9K\omega}{2+\omega}, \quad K = \xi E_1$$
(20)

Then for the composite viscoelastic layer we have:

$$G_{23} = \gamma G_1 + (1 - \gamma)G_2 = \gamma \frac{E_1}{2(1 + v_1)} + (1 - \gamma)\frac{3}{2}K\omega = \gamma E_1 \left[\frac{1}{2(1 + v_1)} + \frac{3(1 - \gamma)}{2\gamma}\frac{K}{E_1}\omega\right]$$

$$\xi = \frac{K}{E_1} \quad \text{then} \quad G_{23} = \gamma E_1 \left[\frac{\gamma + 3(1 - \gamma)\xi\omega(1 + v_1)}{2\gamma(1 + v_1)}\right]$$

$$\frac{1}{G_{12}} = \frac{\gamma}{G_1} + \frac{(1 - \gamma)}{G_2} = \frac{2\gamma(1 + v_1)}{E_1} + \frac{2(1 - \gamma)}{3K\omega} = \frac{\gamma}{E_1} \left[\frac{6\gamma\omega\xi(1 + v_1) + 2(1 - \gamma)}{3\gamma\omega\xi}\right]$$

so from this relation we can write the ratio $k = \frac{s_{66}}{s_{44}}$ in the form

$$k = \frac{G_{23}}{G_{12}} = \gamma^2 \left[\frac{3\gamma^2 \omega \xi (1+v_1) + \gamma (1-\gamma) + 9\gamma \omega^2 \xi^2 (1-\gamma) (1+v_1)^2 + 3\omega \xi (1+v_1) (1-\gamma)^2}{3\gamma^2 \omega \xi (1+v_1)} \right]$$
(21)

where γ is the ratio between the thickness of the reinforcing layer to the thickness of the whole material.

Suppose that $\overline{\sigma}_{r\theta} = \frac{\sigma_{r\theta}}{E_1}, \overline{\sigma}_{\theta z} = \frac{\sigma_{\theta z}}{E_1}$ the relations (6) become after removing the bar symbol

$$\sigma_{r\theta} = -\left[\frac{3\omega\xi}{6\gamma\omega\xi(1+\nu_1)+2(1-\gamma)}\right]_0^{\infty} p A(p) \frac{\sinh\left[\frac{p}{\sqrt{K}}(h-z)\right]}{\sinh\left(\frac{ph}{\sqrt{K}}\right)} J_2(pr)dp$$

$$\sigma_{\theta z} = -\sqrt{\frac{3\omega\xi\gamma+9(1-\gamma)\omega^2\xi^2(1+\nu_1)}{12\gamma\omega\xi(1+\nu_1)^2+4(1+\nu_1)(1-\gamma)}} \int_0^{\infty} p A(p) \frac{\cosh\left[\frac{p}{\sqrt{K}}(h-z)\right]}{\sinh\left(\frac{ph}{\sqrt{K}}\right)} J_1(pr)dp$$

The displacement and stresses may be considered as constant function in elastic composites and operator functions of time in viscoelastic composites. In general, $u_{\theta}, \sigma_{r\theta}$, and $\sigma_{\theta z}$ can be represented according to Illyushin's approximation method of the unified form:

$$f(r,\omega) = \sum_{i=1}^{5} A_i \varphi_i(\omega), \qquad (22)$$

where $f(r, \omega)$ is one of the functions $u_{\theta}, \sigma_{r\theta}$, and $\sigma_{\theta z}$

$$\varphi_{1} = 1 , \quad \varphi_{2} = \omega,$$

$$\varphi_{3} = \pi = \frac{1}{\omega} , \quad \varphi_{4} = g_{1/2}(\omega),$$

$$\varphi_{5} = g_{2}(\omega) , \quad g_{\beta}(\omega) = \frac{1}{1 + \beta\omega} , \quad \beta = \frac{1}{2}, 2$$
(23)

The constants A_i , $(i = 1, \mathbf{K}, 5)$ are constants to be found from the system of linear algebraic equations.

$$L_{ij}A_{j} = B_{i},$$

$$L_{ij} = \int_{0}^{1} \varphi_{i}(\omega) \varphi_{j}(\omega) d\omega,$$

$$B_{i} = \int_{0}^{1} f(r,w) \varphi_{i}(\omega) d\omega, \quad (i, j = 1, \mathbf{K}, 5)$$
(24)

Assuming an exponential relaxation function:

$$\omega = a_1 + b_1 e^{-\alpha t}, \tag{25}$$

where a_1, b_1, α are constants determined experimentally.

The Laplace-Carson transformation can be used to determine the functions $\pi(t)$ and $g_{\beta}(t)\left(\beta = \frac{1}{2}, 2\right)$. Denoting the transformations of $\pi(t), g_{\beta}(t)$ by π^*, g_{β}^* , since $\omega^*(s) = a_1 + \frac{b_1 s}{(s + \alpha)}$,

thus

$$\pi(t) = \frac{1}{a_1} \left\{ 1 - \frac{b_1}{a_1 + b_1} e^{\frac{-a_1 \tau}{a_1 + b_1}} \right\} , \quad \tau = \alpha t ,$$

$$g_{\beta}(t) = \frac{1}{1 + \beta a_1} \left[1 - \frac{\beta b_1}{1 + \beta (a_1 + b_1)} e^{\frac{-(1 + \beta a_1)\tau}{[1 + \beta (a_1 + b_1)]}} \right], \quad \beta = \frac{1}{2}, 2.$$
(26)

The eq.(22) for a viscoelastic composite may be recorded to obtain the explicit formulae for $f(r, \omega)$ as function of (r, t). Thus we have:

$$f(r,t) = A_{1}\Omega(t) + A_{2}\int_{0}^{t}\omega(t-\tau)d\Omega(\tau) + A_{3}\int_{0}^{t}\pi(t-\tau)d\Omega(\tau) + A_{4}\int_{0}^{t}g_{\frac{1}{2}}(t-\tau)d\Omega(\tau) + A_{4}\int_{0}^{t}g_{2}(t-\tau)d\Omega(\tau),$$

$$\Omega(t) = \begin{cases} \Omega_{0}t & 0 \le t \le t_{0} \\ \Omega_{0}h(t-t_{0}) & t \ge t_{0} \end{cases}$$
(27)

where Ω_0 is constant, t_0 is the initial time of rotation, and h(t) is the Heaviside unit step function.

then we have

$$f(r,t) = \sum_{j=1}^{5} A_j X_j$$
 (28)

Where

$$\begin{split} X_{1} &= \Omega_{0} \begin{cases} -F_{1}(1) + 2a_{1}t_{0} & 0 \leq \tau \leq t_{0} \\ a_{1} + F_{2}(1) & \tau \geq t_{0} \end{cases} \\ X_{2} &= \Omega_{0} \begin{cases} \frac{1}{a_{1}^{2}}F_{1}\bigg(\frac{a_{1}}{a_{1} + b_{1}}\bigg) & 0 \leq \tau \leq t_{0} \\ \frac{1}{a_{1}}\bigg(\frac{F_{2}\bigg(\frac{a_{1}}{a_{1} + b_{1}}\bigg)}{(a_{1} + b_{1})}\bigg) & \tau \geq t_{0} \end{cases} \end{split}$$

$$\begin{split} X_{3} &= \Omega_{0} \begin{cases} \frac{2}{\left(2+a_{1}\right)^{2}} \left(F_{1}\left(\frac{2+a_{1}}{2+a_{1}+b_{1}}\right)-2t_{0}\right) & 0 \leq \tau \leq t_{0} \\ \frac{2}{\left(2+a_{1}\right)} \left(1-\frac{F_{2}\left(\frac{2+a_{1}}{2+a_{1}+b_{1}}\right)}{\left(2+a_{1}+b_{1}\right)}\right) & \tau \geq t_{0} \end{cases} \\ X_{4} &= \Omega_{0} \begin{cases} \frac{1}{\left(1+2a_{1}\right)^{2}} \left(2F_{1}\left(\frac{1+2a_{1}}{1+2a_{1}+2b_{1}}\right)-t_{0}\right) & 0 \leq \tau \leq t_{0} \\ \frac{1}{\left(1+2a_{1}\right)} \left(1-\frac{2F_{2}\left(\frac{1+2a_{1}}{1+2a_{1}+2b_{1}}\right)}{\left(1+2a_{1}+2b_{1}\right)}\right) & \tau \geq t_{0} \end{cases} \end{split}$$

Here $F_1(y) = b_1 e^{-y\tau} + a_1 t_0 - b_1 e^{-y(\tau - t_0)}$, $F_2(y) = b_1 e^{-y(\tau - t_0)}$

Numerical Example and Discussion

A numerical example for torsion of anisotropic reinforced viscoelastic layer will be given, the example includes the layer of elastic and viscoelastic form. These numerical computations are carried out for the displacement and stresses that being reported herein:

Computations were carried out for the following values of parameters

$$a=1, \quad h=1.5, \quad v_1=\frac{1}{3}, \quad \xi=\frac{4}{3}, \quad \gamma=\frac{1}{10}, \quad \theta_0=5, \quad t_0=10, \quad a_1=0.1, \quad b_1=0.9,$$

The coefficient α depends on the scale of time parameter and let $\tau (\equiv \alpha t)$

The numerical examples of the stresses and displacement are calculated by using the following data:

material	value of k
Anisotropic	0.5
Isotropic	1
Anisotropic	2

The results of the present investigations are given in tables (1-12). Note that the results are given for different values of geometric, and constitutive parameters.

Figs.(2)-(6) show the stresses and displacement in the dimensionless form:

$$\overline{u} = \frac{u}{\Omega}, \ \overline{\sigma}_{\theta z} = \frac{\sigma_{\theta z}}{\Omega G_{23}}, \ \overline{\sigma}_{r\theta} = \frac{\sigma_{r\theta}}{\Omega G_{12}},$$

and then we will remove the bars symbol.

The variation of displacement u with r for all cases outside the shaft (z=0) for different values of constitutive parameter k is illustrated in Fig.(2). The radial distribution of stresses, $\sigma_{\theta z}$, on the contact surface under shaft $(z=0,0 \le r \le a)$ are plotted in Fig.(2). The variation of $\sigma_{\theta z}$ on the contact surface is shown in Fig.3 and at the fixed base (z=h) is shown in Fig.(4). Also, the $\sigma_{r\theta}$ on the contact surface (z=0) is plotted in Fig.(3,5) and also the variation of stresses $\sigma_{\theta z}$ and

 $\sigma_{r\theta}$ and displacement u at the middle surface of the layer $\left(z = \frac{h}{2}\right)$ in Figs (6).

It seens from Fig.(2) that displacement u attained its maximum value at the points of the edge of the shaft and then decreases very rapidly. The magnitude of $\sigma_{\theta z}$ is slightly increases when kdecreases and it increases with r to attained maximum value at the boundary of the shaft r=a, and it vanishes outside of the shaft. The influence of ratio $\frac{h}{a}$ (thickness of layer/radius of shaft) to the distribution and magnitude is relatively insignificant. The stress $\sigma_{\theta z}$ on the fixed base surface increases with r and attained it maximum at the boundary of the circle r=r and the decreases and vanishes at infinity. As shown in Fig.(5), the stress $\sigma_{r\theta}$ on the contact surface z=0 takes the same behavior but with discontinuity at r=at, This continuity shon also for all z-levels as shown in Fig.6 which illustrate also the behavior of stress $\sigma_{\theta z}$ and displacement u.

It can be seen from Fig.(6) that the displacement u_{θ} at $z = \frac{h}{2}$ has a maximum value on the contact surface under the edge of the shaft and then decrease rapidly towards the centre of the shaft and outside the shaft. Also in Fig.(6) the stresses $\sigma_{\theta z}$, $\sigma_{r\theta}$ has the same behavior of the

displacement at $z = \frac{h}{2}$.

Tables (1-7) give the same values of coefficient A_i , i = 1, 2, 3, 4, 5 given in eq.(28) for the $u_{\theta}, \sigma_{r\theta}, \sigma_{\theta z}$ in terms of radius *r*, from which we have the viscoelastic solution at different values of *r* as explicit functions of the time.

Figs.(7)-(19) show the type of relaxation with time that occurs in the $u_{\theta}, \sigma_{r\theta}, \sigma_{\theta z}$ respectively at some particular points of the disk, from which we see that all values are start decrease with time, and then increase to attend their asymptotic values, meaning that a steady state established in the layer.

Conclusion and Recommendation

Based on the above illustration and the numerical example-real-world application, we have concluded that the anisotropic layer whose material has three mutually perpendicular directions of elasticity symmetry parallel to the axis of coordinates can be solved using innovative scheme for formal solution of the stress and displacement. This new framework of proposes form a solution of the stress and displacement, which is occurring in an infinite anisotropic reinforced viscoelastic layer, when it is twisted by means of turning a rigid cylindrical shaft attached to it. Therefore, by modifying the problem with this additional important terms and assuming that the layer is made of composite viscoelastic material is accomplished. In addition, we find the distributions of stresses and displacement in the proposed model. Furthermore, the behavior of the layer is governed by the equilibrium equation are solved by means of Hankel transforms. This solution of purely elastic layer is obtained, and then the problem of model with anisotropic reinforced viscoelastic layer is solved using the correspondence principle and Ilyushin's approximation method.

Appendix:

a)
$$\begin{cases} B(k) = \sum_{i=1}^{3} A_{i-1} \left(\sum_{q=0}^{\infty} (-1)^{q} \sum_{m=0}^{\infty} u_{k}(i,1) - \sum_{q=0}^{\infty} (-1)^{q} \sum_{m=0}^{\infty} u_{k}(i,2) \right) & k = 1,2,3 \\ B_{1}(k) = \sum_{i=1}^{3} H_{i-1} \left(\sum_{q=0}^{\infty} (-1)^{q} \sum_{m=0}^{\infty} u_{k}(i,1) - \sum_{q=0}^{\infty} (-1)^{q} \sum_{m=0}^{\infty} u_{k}(i,2) \right) & k = 4,5,6 \end{cases}$$

and

$$\mathbf{b} \qquad \begin{cases} u_1(i,j) = \frac{u_0 a^{2i} \Gamma(2i+1+2m)}{\left(a^2+c_j^2\right)^{\left(\frac{2i+1}{2}+m\right)}} \,_2F_1\left(\frac{2i-1}{2}-m,\frac{2i+1}{2}+m;\frac{4i+1}{2};\frac{a^2}{a^2+c_j^2}\right) \\ u_2(i,j) = \frac{u_0 a^{2i} \Gamma(2i+2+2m)}{\left(a^2+c_j^2\right)^{\left(i+1+m\right)}} \,_2F_1\left(i-1-m,i+1+m;\frac{4i+1}{2};\frac{a^2}{a^2+c_j^2}\right) \\ u_3(i,j) = \frac{\left(\frac{r}{2}\right) u_2(i,j)}{\left(2+m\right)} \end{cases}$$

$$\mathbf{c} \qquad \begin{cases} u_4(i,j) = \frac{(-1)^m}{m!} \frac{\left(\frac{a}{2}\right)^{(2i+2m)} \Gamma\left(2i+1+2m\right)}{\Gamma\left(\frac{2i+1}{2}+m\right) \left(r^2+c_j^2\right)^{\left(\frac{2i+1}{2}+m\right)}} {}_2F_1\left(i-1-m,\frac{2i+1}{2}+m;2;\frac{r^2}{r^2+c_j^2}\right) \\ u_5(i,j) = \frac{(-1)^m \left(\frac{a}{2}\right)^{(2i+2m)} \Gamma\left(2i+2+2m\right)}{m! \Gamma\left(\frac{4i+1}{2}+m\right) \left(r^2+c_j^2\right)^{\left(i+1+m\right)}} {}_2F_1\left(i+1+m,-\frac{2i-1}{2}-m;2;\frac{r^2}{r^2+c_j^2}\right) \\ u_6(i,j) = \frac{(-1)^m \left(\frac{a}{2}\right)^{(2i+2m)} \Gamma\left(2i+3+2m\right)}{m! \Gamma\left(\frac{4i+1}{2}+m\right) \left(r^2+c_j^2\right)^{\left(\frac{2i+3}{2}+m\right)}} {}_2F_1\left(1-i-m,\frac{2i+3}{2}+m;3;\frac{r^2}{r^2+c_j^2}\right) \end{cases}$$

where

$$A_{0} = \frac{H_{0}}{3}, \quad A_{1} = \frac{H_{1}}{35}, \quad and \quad A_{2} = \frac{H_{2}}{5465}$$

$$c_{1} = 2q \ b + \frac{z}{\sqrt{k}}, \qquad c_{2} = 2q \ b + 2b - \frac{z}{\sqrt{k}}$$

$$u_{0} = \frac{(-1)^{m} \left(\frac{r}{2}\right)^{(1+2m)}}{m! \Gamma(2+m)}, \quad i = 1, 2, 3 \text{ and } j = 1, 2$$

d) For 0 < b < a

$$\int_{0}^{\infty} \frac{J_{\alpha-\beta}(at)J_{\gamma-1}(bt)}{t^{\gamma-\alpha-\beta}} dt = \sum_{m=0}^{\infty} \frac{(-1)^m b^{\gamma+2m-1} \Gamma(2\alpha+2m) \Gamma(\frac{1}{2})}{m! \Gamma(\gamma+m) 2^{\alpha-\beta+\gamma+2m-1} a^{\alpha+\beta+2m} \Gamma(1-\beta-m) \Gamma(\alpha+m+\frac{1}{2})}$$

e) For
$$|z| < \sqrt{a^2 + c^2}$$

$$\int_{0}^{\infty} e^{-ct} \frac{J_{\alpha-\beta}(at)J_{\gamma-1}(zt)}{t^{\gamma-\alpha-\beta}} dt = \sum_{m=0}^{\infty} \frac{(-1)^m (\frac{z}{2})^{\gamma+2m-1} (\frac{a}{2})^{\alpha-\beta} \Gamma(2\alpha+2m)}{m! \Gamma(\gamma+m) (a^2+c^2)^{\alpha+m} \Gamma(\alpha-\beta+1)} {}_2F_1 \left(\alpha+m, \frac{1}{2}-\beta-m; \alpha-\beta+1; \frac{a^2}{a^2+c^2}\right)$$



Fig.1. Infinite anisotropic layer subjected to a twisting moment.



Fig. 2. Distribution of $u_{\theta}(r,0)$ in r direction.



Fig.3. Distribution of $\sigma_{\theta_z}(r,0)$ when $0 \le r \le a$.



Fig.5. Distribution of $\sigma_{\theta r}(r,0)$ in r direction.



Fig.4. Distribution of $\sigma_{\theta z}(r,h)$ with r.



Fig. 6. Stresses and displacement at $z = \frac{h}{2}, k = 1$.



$$\left(r=0.3, z=\frac{h}{2}\right)$$
 with time.

Fig. 12. Variation of stress and displacement at

 $\sigma_{\theta r}$

u_θ

 $\sigma_{\theta z}$

5

5

$$\left(r=3, z=\frac{h}{2}\right)$$
 with time.

r	A ₁	A ₂ x10 ⁻⁴	A ₃ x10 ⁻⁷	A ₄ x10 ⁻³	A ₅ x10 ⁻⁴
0	0	0	0	0	0
0.1	0.18282-	0.39777-	-0.4917	-0.1998	0.4508
0.2	-0.3711	-1.2897	-1.5876	-0.6476	1.4598
0.3	-0.5726	0.02612	0.01842	0.02431	-0.0254
0.4	-0.7949	0.48497	0.61916	0.24446	-0.5549
0.5	-1.0521	2.42826	3.00983	1.22012	-2.7541
0.6	-1.3632	-5.0836	-6.3114	-2.5549	5.7689
0.7	-1.7851	5.07556	0.06379	2.55443	-5.7818
0.8	-2.4208	5.47219	6.67828	2.74492	-6.1778
0.9	-3.6288	15.1039	18.5393	7.58186	-17.084
1	-6.711	16.2087	19.9344	8.13715	-18.34

Table (2) Values of Ai for $\sigma_{\theta z}(r,0)$.

r	$A_1 \ x10^{-2}$	A ₂ x10 ⁻⁷	A ₃ x10 ⁻⁸	A ₄ x10 ⁻⁶	A ₅ x10 ⁻⁷
0	0	0	0	0	0
0.1	9.9886	135.81	1.6886	68.264	-154.18
0.2	19.985	177.12	2.0983	88.574	-198.28
0.3	30.023	-666.69	-8.3581	-335.39	758.657
0.4	40.004	149.63	1.8734	75.227	-170.02
0.5	50.098	-1574.8	-19.691	-792.12	1791.11
0.6	60.022	1276.6	15.987	642.13	-1452.2
0.7	70.168	-1197.6	-15.04	-602.62	1363.58
0.8	80.153	588.54	7.4085	296.19	-670.46
0.9	90.206	716.74	8.8396	359.9	-811.51
1	100.22	1782.7	21.998	895.36	-2019.4
1	78.539	-1210.1	-14.795	-607.09	1366.73
2	8.0591	-3.1576	-0.0382	-1.575	3.51924
3	1.8526	-95.973	-1.1742	-48.153	108.419
4	0.4719	18.575	0.2301	9.3326	-21.063
5	0.1266	4.6538	0.0575	2.3377	-5.274
6	0.0356	0.5852	0.0072	0.2939	-0.6629
7	0.011	0.128	0.0016	0.0643	-0.145
8	0.0043	0.0878	0.0011	0.044	-0.0991
9	0.0025	-0.1078	-0.0013	-0.0542	0.12216
10	0.0021	0.0365	0	0.0184	-0.0416

Table (1) Values of Ai for u(r,0).

r	\mathbf{A}_{1}	A ₂ x10 ⁻⁵	A ₃ x10 ⁻⁸	A ₄ x10 ⁻⁵	A ₅ x10 ⁻⁵
0	0	0	0	0	0
0.1	-0.0032	-0.1101	-0.1324	-0.5512	0.12372
0.2	-0.013	0.2827	0.3456	1.4184	-0.3194
0.3	-0.0298	-1.1252	-1.4279	-5.6678	1.28487
0.4	-0.0544	-1.3258	-1.6408	-6.6607	1.50301
0.5	-0.0884	2.24	2.774	11.254	-2.5398
0.6	-0.1336	-3.486	-4.3388	-17.524	3.95857
0.7	-0.1944	-11.265	-13.925	-56.586	12.7657
0.8	-0.2807	8.8021	10.898	44.222	-9.98
0.9	-0.4123	21.465	27.027	108.04	-24.459
1	-0.6556	20.845	25.575	104.63	-23.573
1	-4.852	-161.49	-196.99	-810.04	182.293
2	-0.169	10.622	13.208	53.392	-12.059
3	-0.032	-1.6032	-1.9757	-8.0507	1.81527
4	-0.0075	0.1329	0.1637	0.6673	-0.1505
5	-0.0019	0.0474	0.0586	0.2379	-0.0537
6	-0.0005	0.0052	0.0065	0.0263	-0.0059
7	-0.0001	-0.0041	-0.0051	-0.0207	0.00467
8	-4E-05	0.0001	0	0.0007	-0.0002
9	-1E-05	0.0002	0	0.0012	-0.0003
10	0	0	0	-0.0003	0.00005

Table (3) Values of Ai for $\,\,\sigma_{ heta z}\left(r,h
ight)$

r	A ₁	A ₂ x10 ⁻⁵	A ₃ x10 ⁻⁷	A ₄ x10 ⁻⁵	A ₅ x10 ⁻⁵
0	0	0	0	0	0
0.1	0.02987	1.16902	0.14541	5.87594	-1.3272
0.2	0.05897	1.02107	0.12994	5.14531	-1.1672
0.3	0.08648	-0.8583	-0.1044	-4.3033	0.9678
0.4	0.11131	1.23207	0.15093	6.18272	-1.3925
0.5	0.13298	-3.1047	-0.3817	-15.586	3.5129
0.6	0.14977	7.60472	0.93883	38.1941	-8.6147
0.7	0.16218	1.44551	0.18431	7.28641	-1.6537
0.8	0.16849	7.85152	0.98488	39.5049	-8.9379
0.9	0.16937	8.41702	1.04244	42.2896	-9.5444
1	0.16526	5.25937	0.64145	26.3782	-5.9355
1	0.16544	-2.9145	-0.3689	-14.679	3.3267
2	0.05302	-2.0036	-0.2491	-10.071	2.2744
3	0.013	-0.4798	-0.0593	-2.41	0.5437
4	0.3341	-0.073	-0.0091	-0.3669	0.0829
5	0.00089	-0.0099	-0.0013	-0.0499	0.0113
6	0.00025	-0.0072	-0.0009	-0.036	0.0082
7	0.00007	0.00144	0.00018	0.00724	-0.0016
8	0.00002	-0.0018	-0.0002	-0.0092	0.0021
9	0.00001	0.00029	0.00003	0.00147	-0.0003
10	0.00001	0.00003	0	0.00019	-4E-05

Table (4) Values of Ai for $\,\sigma_{r heta}(r,0)$.

r	\mathbf{A}_{1}	A ₂ x10 ⁻⁵	A ₃ x10 ⁻⁷	A4 x10 ⁻⁵	A ₅ x10 ⁻⁵
0	0	0	0	0	0
0.1	0.03	1.0199	0.1259	5.1223	-1.1554
0.2	0.0592	-1.8672	-0.2317	-9.3831	2.11839
0.3	0.0866	2.6432	0.3284	13.285	-3
0.4	0.1117	-1.0766	-0.1327	-5.4064	1.21914
0.5	0.1333	-1.6516	-0.2053	-8.3019	1.87506
0.6	0.1505	-1.6353	-0.1971	-8.1915	1.83918
0.7	0.1627	-0.9382	-0.1205	-4.7331	1.0757
0.8	0.1693	-2.4083	-0.3033	-12.123	2.74484
0.9	0.1697	10.298	1.2737	51.734	-11.673
	0.166	-3.6165	-0.4494	-18.176	4.10472
1	0.1657	0.7714	0.0909	3.8549	-0.8619
2	0.0532	-2.1474	-0.2686	-10.801	2.44241
3	0.013	-0.4986	-0.0614	-2.5032	0.56428
4	0.0034	-0.191	-0.0238	-0.9602	0.21688
5	0.0009	-0.0038	-0.0005	-0.0189	0.00424
6	0.0003	0.0121	0.0015	0.0609	-0.0137
7	8E-05	-0.0006	-7E-05	-0.003	0.00067
8	3E-05	-0.0016	-0.0002	-0.0079	0.00178
9	1E-05	0.0005	6E-05	0.0023	-0.0005
10	1E-05	-0.0002	-3E-05	-0.0011	0.00024

A₃ x10⁻⁷

0

-0.0061

-0.0077

0.033

0.0912

-0.1983

-0.0684

-0.3288

-0.2063

0.0069

-0.0761

1.9738

-0.2418

-0.056

-0.0179

-0.0045

-0.0017

-0.0002

9E-05

-1E-05

A4 x10⁻⁵

0

-0.2454

-0.3145

1.3476

3.6696

-8.0763

-2.7892

-13.354

-8.4458

0.2722

-3.1681

79.887

-9.8784

-2.3247

-0.7335

-0.1836

-0.0672

-0.006

0.0039

-0.0006

A₅ x10⁻⁵

0

0.05544

0.07096

-0.3037

-0.8295

1.8211

0.62878

3.01311

1.90242

-0.0619

0.71089

-18.039

2.22597

0.52207

0.16521

0.04137

0.01518

0.00137

-0.0009

0.00013

-0.0001

Table (5) Values of Ai for	$u\left(r,\frac{h}{2}\right)$
----------------------------	-------------------------------

-0.6327

15.896

-1.968

-0.464

-0.1462

-0.0366

-0.0134

-0.0012

0.0008

-0.0001

-3E-05 0.00188 0.00023 0.00942 -1E-05 -0.0014 -0.0002 -0.007-1E-05 0.00005 0 0.00259

A₃ x10⁻⁷

0

0.05439

-0.1116

0.80469

1.15049

-1.5828

1.39095

-1.1103

2.50159

-1.001

2.37293

-1.5121

0.17523

-0.0196

-0.0072

-0.0008

-0.0005

0.00018

A4 x10⁻⁵

0

2.30848

-4.7073

32.3774

46.6951

-65.902

55.671

-43.95

101.374

-40.014

95.4451

-60.893

7.19988

-0.8201

-0.2922

-0.0317

-0.0186

0.00742

A₅ x10⁻⁵

0

-0.5159

1.0537

-7.3198

-10.536

14.792

-12.6

9.9709

-22.885

9.0602

-21.581

13.763

-1.6206

0.1839

0.0659

0.0071

0.0042

-0.0017

-0.0021

0.0016

-0.0006

A₂ x10⁻⁵

0

0.46214

-0.9414

6.43789

9.29524

-13.159

11.0627

-8.7213

20.1745

-7.9492

18.9767

-12.11

1.43524

-0.1638

-0.0582

-0.0063

-0.0037

0.00148

r

0

0.1

0.2

0.3

0.4

0.5

0.6

0.7

0.8

0.9

1

2

3

4 5

6

7

8

9

10

 $A_1 \\$

0

-0.0947

-0.1875

-0.277

-0.36

-0.4325

-0.4931

-0.5335

-0.553

-0.5454

-0.5179

-0.3249

-0.0632

-0.0138

-0.0035

-0.0009

-0.0003

-8E-05

A ₂ x10 ⁻⁵	A ₁	r
0	0	0
5 -0.0488	-0.0015	0.1
8 -0.0626	-0.0058	0.2
9 0.2684	-0.0129	0.3
6 0.7297	-0.0226	0.4
4 -1.6083	-0.0344	0.5
-0.5555	-0.048	0.6
5 -2.6584	-0.0625	0.7
1 -1.6829	-0.0771	0.8
3 0.0539	-9.0583	.9

1

2

3

4

5

6

7

8

9

-0.1019

-0.2352

-0.1011

-0.0223

-0.0053

-0.0013

-0.0004

-9E-05

-2E-05

0

Table (6) Values of Ai for $\sigma_{\theta z}\left(r,\frac{h}{2}\right)$

10	0	0.0001	1E-05	0.0006
	Table ((7) Values	of A _i for	$\sigma_{r\theta}\left(r,\frac{h}{2}\right)$

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