Bounds for the Zeros of Polynomials Based on Spectral Norm Inequalities

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ABSTRACT

In this paper we present new estimates for the spectral norm for the square of the Frobenius companion matrix. We apply these estimates to get new bounds for the zeros of polynomials and commutator inequalities.

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1. Introduction

The Frobenius companion matrix plays an important link between matrix analysis and polynomials. It is used for the location for the zeros of polynomials by matrix method, see [3] and [12]. It is also used for the numerical approximation, see [14].

Many papers appeared to give new bounds for the zeros of polynomials or to improve some classical bounds, see, e.g., [1], [2], [5], [7]-[11], [16], and [17]. Fujii and Kubo [1] used the companion matrix to give new proofs of some classical bounds.

In the presented paper, we will give new estimates for the spectral norm for the square of the Frobenius companion matrix. These estimates based on spectral norm inequalities for partitioned matrices. We apply these estimates to get new bounds for the zeros of polynomials and commutator inequalities. Section 2 presents some classical and recent bounds such as Cauchy's bound, Montel's Bound, Carmichael-Mason's bound, s_1-bound, and FK-bound.

2. Classical and recent bounds for the zeros of polynomials

This section introduces some classical and recent bounds for the zeros of polynomials, which can be proved with the aid of the companion matrix.

The Frobenius companion matrix of the monic polynomial

$$p(z) = z^n + a_n z^{n-1} + a_{n-1} z^{n-2} + \cdots + a_2 z + a_1,$$

$n \geq 2$, is

$$C(p) = \begin{bmatrix} -a_n & -a_{n-1} & \ldots & -a_2 & -a_1 \\ 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & 0 \end{bmatrix}.$$

The characteristic polynomial of $C(p)$ is $p(z)$. Thus, the eigenvalues of $C(p)$ are the zeros of $p(z)$. For simplicity we write C instead of $C(p)$.

The following theorem contains classical bounds for the zeros of polynomials and these bounds can be found in [3].

Theorem 2.1: If z is any zero of p , then

1. $$|z| \leq \max\{ |a_1|, 1+|a_2|, 1+|a_3|, \ldots, 1+|a_n| \} \leq 1+\max\{ |a_1|, |a_2|, |a_3|, \ldots, |a_n| \}.$$
(Cauchy's bound)
2. $$|z| \leq \max\{1, |a_1|+|a_2|+|a_3|+\ldots+|a_n|\} \leq 1+|a_1|+|a_2|+|a_3|+\ldots+|a_n|.$$
(Montel's bound)
3. $$|z| \leq \left(1+|a_1|^2+|a_2|^2+|a_3|^2+\ldots+|a_n|^2\right)^{\frac{1}{2}}.$$
(Carmichael-Mason's bound)

For other classical bounds for the zeros of polynomials see [12], [13], and [15].

Theorem 2.2: If z is any zero of p, then

1. $$|z| \le \cos\left(\frac{\pi}{n+1}\right) + \frac{1}{2}\left(|a_n| + \left(\sum_{j=1}^{n} |a_j|^2\right)^{\frac{1}{2}} \right).$$

(FK-bound [2])

2. $$|z| \le \left(\frac{1}{2}\left(1+\sum_{j=1}^{n} |a_j|^2 + \sqrt{\left(1+\sum_{j=1}^{n} |a_j|^2\right)^2 - 4|a_1|^2}\right)\right)^{\frac{1}{2}}.$$

(s_1-bound [5])

3. $$|z| \le \left(\frac{1}{2}\left(\delta + 1 + \sqrt{(\delta-1)^2 + 4\,\delta'}\right)\right)^{\frac{1}{4}},$$

(Kitt-bound [7])

where

$$\delta = \frac{1}{2}\left(\alpha + \beta + \sqrt{(\alpha-\beta)^2 + 4|\gamma|^2}\right),$$

$$\delta' = \frac{1}{2}\left(\alpha' + \beta' + \sqrt{(\alpha'-\beta')^2 + 4|\gamma'|^2}\right),$$

$$\alpha = \sum_{j=1}^{n} |a_j|^2, \beta = \sum_{j=1}^{n} |b_j|^2, \alpha' = \sum_{j=3}^{n} |a_j|^2,$$

$$\beta' = \sum_{j=3}^{n} |b_j|^2, \gamma = -\sum_{j=1}^{n} \bar{a}_j b_j \text{ , and}$$

$$\gamma' = -\sum_{j=3}^{n} \bar{a}_j b_j .$$

3. Estimates for the spectral norm of the square of the companion matrix

In this section, we will give new bounds for the zeros of polynomials depending on estimates of $\|C^2\|$. In this section, z stands for a zero of p and

$$K = C^2 = \begin{bmatrix} b_n & b_{n-1} & \cdots & b_3 & b_2 & b_1 \\ -a_n & -a_{n-1} & \cdots & -a_3 & -a_2 & -a_1 \\ 1 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 \end{bmatrix},$$

where $b_j = a_n a_j - a_{j-1}, \quad j = 1, 2, \ldots, n$, with $a_0 = 0$.

The following lemma can be easily proved.

Lemma 3.1: The singular values of $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ are

$$s_1 = \left[\frac{1}{2}\left(|a|^2+|b|^2+|c|^2+|d|^2+\sqrt{\left(|a|^2+|b|^2-|c|^2-|d|^2\right)^2+4|a\bar{c}+b\bar{d}|^2}\right)\right]^{\frac{1}{2}}$$

$$= \left[\frac{1}{2}\left(|a|^2+|b|^2+|c|^2+|d|^2+\sqrt{\left(|a|^2+|c|^2-|b|^2-|d|^2\right)^2+4|a\bar{b}+c\bar{d}|^2}\right)\right]^{\frac{1}{2}},$$

$$s_2 = \left[\frac{1}{2}\left(|a|^2+|b|^2+|c|^2+|d|^2-\sqrt{\left(|a|^2+|b|^2-|c|^2-|d|^2\right)^2+4|a\bar{c}+b\bar{d}|^2}\right)\right]^{\frac{1}{2}}$$

$$= \left[\frac{1}{2}\left(|a|^2+|b|^2+|c|^2+|d|^2-\sqrt{\left(|a|^2+|c|^2-|b|^2-|d|^2\right)^2+4|a\bar{b}+c\bar{d}|^2}\right)\right]^{\frac{1}{2}}.$$

The following lemma is basic in our work and can be found in [4].

Lemma 3.2: Let $A \in M_n(\mathbb{C})$ be partitioned as

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

where A_{ij} is an $n_i \times n_j$ matrix for $i, j = 1, 2$, with $n_1 + n_2 = n$. If

$$\tilde{A} = \begin{bmatrix} \|A_{11}\| & \|A_{12}\| \\ \|A_{21}\| & \|A_{22}\| \end{bmatrix},$$

then $\|A\| \le \|\tilde{A}\|$.

The following lemma was given by the author in [8].

Lemma 3.3 : Let

$$L = \begin{bmatrix} -a_{n-1} & -a_{n-2} & \cdots & -a_3 & -a_2 & -a_1 \\ 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 \end{bmatrix},$$

with $n \ge 4$. Then

$$\|L\|^2 = \frac{1}{2}\left(1+\alpha+\sqrt{(1+\alpha)^2 - 4\left(|a_1|^2+|a_2|^2\right)}\right),$$

where $\alpha = \sum_{j=1}^{n-1} |a_j|^2$.

Theorem 3.4: If $n \ge 4$, then

$$\|K\|^2 \le \frac{1}{2}\left[\alpha + \beta + \sqrt{(\alpha-\beta)^2 + 4\gamma^2}\right],$$

where $\alpha = 1 + |a_n|^2 + |b_n|^2$,

$$\beta = \|K_{12}\|^2 + \|K_{22}\|^2, \|K_{12}\|^2 = \sum_{j=1}^{n-1} |b_j|^2,$$

$$\| K_{22}\|^2 = \frac{1}{2}\left[1+\sum_{j=1}^{n-1}|a_j|^2+\sqrt{\left(1+\sum_{j=1}^{n-1}|a_j|^2\right)^2-4\left(|a_1|^2+|a_2|^2\right)}\right],$$

and $\gamma = \| K_{12}\| |b_n| + \| K_{22}\| \sqrt{1+|a_n|^2}$.

Proof: Let $K_{12} = [b_{n-1} \ \ \dots \ \ b_1]$,

$K_{21} = [-a_n \ \ 1 \ \ 0 \ \ \dots \ \ 0]^T$, and

$$K_{22} = \begin{bmatrix} -a_{n-1} & \dots & -a_3 & -a_2 & -a_1 \\ 0 & \dots & 0 & 0 & 0 \\ 1 & \dots & 0 & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \dots & 1 & 0 & 0 \end{bmatrix}.$$

Then $K = \begin{bmatrix} [b_n] & K_{12} \\ K_{21} & K_{22} \end{bmatrix}$. Using Lemma 3.3, we get

$$\| K_{22}\|^2 = s_1^2(K_{22}) = \frac{1}{2}\left(1+\alpha+\sqrt{(1+\alpha)^2-4\left(|a_1|^2+|a_2|^2\right)}\right),$$

where $\alpha = \sum_{j=1}^{n-1}|a_j|^2$, $\|K_{12}\|^2 = \sum_{j=1}^{n-1}|b_j|^2$ and $\|K_{21}\|^2 = 1+|a_n|^2$. Now using Lemma 3.2, we have $\|K\|^2 \le \left\| \begin{bmatrix} |b_n| & \|K_{12}\| \\ \sqrt{1+|a_n|^2} & \|K_{22}\| \end{bmatrix} \right\|^2$.

From Lemma 3.1.9, we have

$$\left\| \begin{bmatrix} |b_n| & \|K_{12}\| \\ \sqrt{1+|a_n|} & \|K_{22}\| \end{bmatrix} \right\|^2 = \frac{1}{2}\left[\alpha+\beta+\sqrt{(\alpha-\beta)^2+4\gamma^2}\right].$$

■

It is well known that

$$|z| \le \| C^2\|^{\frac{1}{2}} \le \| C\|, \qquad (3.1)$$

where z is any zero of p. So the previous estimate of $\| C^2\|$ gives the following bound for the zeros of p.

Corollary 3.5: If $n \ge 4$, then

$$|z|^4 \le \frac{1}{2}\left[\alpha+\beta+\sqrt{(\alpha-\beta)^2+4\gamma^2}\right].$$

bound(1)

Remark 3.6: Let $p(z) = z^4+z^3+z^2+z+100$. Then bound(1) = 13.1392, FK-bound = 51.3165, s_1-bound = 100.015, and Cauchy-bound = 100. While for $p(z) = z^4+9z^3+1$, bound(1) = 10.2064, FK-bound = 9.83671, s_1-bound = 9.10977, and Cauchy-bound = 10. So bound (1) is incomparable with any of these bounds.

Theorem 3.7:

$$\| K\|^2 \le 1+\frac{1}{2}\left[\alpha^2+\beta^2+\sqrt{\left(\alpha^2+\beta^2\right)^2+4\left(2\alpha\beta+1\right)}\right],$$

where

$$\alpha = \left(\frac{1}{2}\left[\sum_{j=n-1}^{n}\left(|a_j|^2+|b_j|^2\right)+\sqrt{\left(\sum_{j=n-1}^{n}\left(|a_j|^2-|b_j|^2\right)\right)^2+4\left|\sum_{j=n-1}^{n}a_j\bar{b}_j\right|^2}\right]\right)^{\frac{1}{2}}$$

and

$$\beta = \left(\frac{1}{2}\left[\sum_{j=1}^{n-1}\left(|a_j|^2+|b_j|^2\right)+\sqrt{\left(\sum_{j=1}^{n-1}\left(|a_j|^2-|b_j|^2\right)\right)^2+4\left|\sum_{j=1}^{n-1}a_j\bar{b}_j\right|^2}\right]\right)^{\frac{1}{2}}$$

Proof: Let $K_{11} = \begin{bmatrix} b_n & b_{n-1} \\ -a_n & -a_{n-1} \end{bmatrix}$,

$K_{12} = \begin{bmatrix} b_{n-2} & \cdots & b_1 \\ -a_{n-1} & \cdots & -a_1 \end{bmatrix}$,

$K_{21} = \begin{bmatrix} I_2 \\ 0 \end{bmatrix}_{(n-2)\times 2}$, and

$K_{22} = \begin{bmatrix} 0 & 0 \\ I_{n-4} & 0 \end{bmatrix}_{(n-2)\times(n-2)}$. Then

$K = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix}$. But $\|K_{21}\| = \|K_{22}\| = 1$.

Some computations give $\|K_{11}\| = \alpha$, $\|K_{12}\| = \beta$. Now using Lemma 3.2, we have

$\|K\|^2 \le \left\| \begin{bmatrix} \|K_{11}\| & \|K_{12}\| \\ 1 & 1 \end{bmatrix} \right\|^2$. Thus,

$$\| K\|^2 \le 1+\frac{1}{2}\left[\alpha^2+\beta^2+\sqrt{\left(\alpha^2-\beta^2\right)^2+4(\alpha\beta+1)^2}\right]$$

and so

$$\| K\|^2 \le 1+\frac{1}{2}\left[\alpha^2+\beta^2+\sqrt{\left(\alpha^2+\beta^2\right)^2+4\left(2\alpha\beta+1\right)}\right].$$

■

Theorem 3.7 and the inequality (3.1) give the following bound for the zeros of p.

Corollary 3.8:

$$|z|^4 \le 1+\frac{1}{2}\left[\alpha^2+\beta^2+\sqrt{\left(\alpha^2+\beta^2\right)^2+4(2\alpha\beta+1)}\right].$$

bound(2)

Remark 3.9: Let $p(z) = z^5+z^4+100$. Then bound(2) = 12.7206, FK-bound = 51.3685, s_1-bound = 100.005, and Cauchy-bound = 100.

While for
$p(z) = z^5 + (100+5i)z^4 + (100+5i)z^3 + 1$,
bound(2) = 131.558, FK-bound = 121.729, and Cauchy-bound = 101.1249. So bound(2) is incomparable with any of these bounds.

***Remark 3.10*:** Our new bounds presented here locate the zeros of p inside discs. The zeros of p can be located inside annuli in those discs by applying these bounds to the polynomial $q(z) = \frac{z^n}{a_1} p\left(\frac{1}{z}\right)$.

***Remark 3.11*:** Our new estimates for $\|C^2\|$ enables us to obtain a lower bound for the spectral norm of the self-commutator of C in terms of the coefficients of p. It has been shown in [6] that

$$\|C^*C - CC^*\| \geq \|C\|^2 - \|C^2\|,$$

where $\|C\|$ was evaluated in [5] and $\|C^2\|$ is estimated in this paper.

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