Exponential Finite Difference Method Applied to Korteweg-de Vries – Burger Equation for Small Times

Ekhlass Saadalluh Al-Rawi
University of Mosul, IRAQ
Email: ekhlass70@yahoo.com

Al-Moatasem Abdul Muhsen Al-Baker
University of Mosul, IRAQ
Email:esamabd23@yahoo.com

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ABSTRACT

Numerical solution of Korteweg-de Vries–Burger (KdV-B) equation is presented using the exponential finite-difference technique. The accuracy of computed solutions is examined by comparison with classical explicit finite-difference and analytical solutions using example .The close results agreement between the current results and the exact solutions confirms that the proposed finite-difference .procedure is an effective technique for the solution of the Korteweg-de Vries–Burger equation at the small times .The proposed method can be applied to partial differential equation without any need for restrictive assumptions on the boundary conditions.

Keywords: Korteweg-de Vries-Burgers equation; Exponential finite-differences; Classical finite-difference method

1. Introduction:

Many researchers have studied Korteweg-de Vries–Burger (KdV-B) equation because of its importance in various physical phenomena. KdV-B equation models the dispersion, dissipation, and nonlinearity [3].

It is well known that many physical phenomena can be described by the Korteweg-de Vries–Burgers equation. Typical examples are provided by the behavior of long waves in shallow water and waves in plasmas. It can serve as a non-linear wave mode of a fluid in an elastic tube, of a liquid with small bubbles and turbulence. This equation is a one-dimension generalization of the model description of the density and velocity fields that takes into account pressure forces as well as the viscosity and the dispersion. It may be a more flexible tool for physicists than Burger's equation. [6]

In [10] we considered the problem of global exponential stabilization by boundary feedback for the Korteweg-de Vries–Burgers equation on the domain [0,1]. In [1] we proposed a more aggressive control low that achieves better performance and proved the existence and stability of solutions of the resulting boundary controlled KdV-B equations. In [4] he presented a relatively new decomposition method to find the explicit and numerical solutions of the KdV equation, Burger's equation and KdV-B equation for the initial conditions. In [6] we devoted to the study the KdV-B equation and applied the finite difference with variable mesh and the semi-analytic Adomain decomposition method. In [3] he presented numerical solutions of Korteweg-de Vries–Burgers equation using modified Bernstein polynomials.

In this paper, we apply the exponential finite difference technique to solve the KdV-B equation. We compared this method with the classical finite difference method as well as the analytical solution.

2. Mathematical formulation

We consider the following form of KdV-B equation

$$\frac{\partial u}{\partial t}+m_1 u\frac{\partial u}{\partial x}+m_2\frac{\partial^2 u}{\partial x^2}+m_3\frac{\partial^3 u}{\partial x^3}=0 \qquad (1)$$

where m_1, m_2, m_3 are constant coefficient with initial condition

$$u(x,0)=u_0(x) \qquad (2)$$

where $u=u(x,t)$ is sufficiently smooth function , and $u_0(x)$ is bounded. The second term of equation (1) describes nonlinearity, the third term corresponds to dissipation, and the last term represents the dispersion. As limiting cases, KdV-B equation reduces to KdV equation when $m_2 \to 0$ and Burgers equation when $m_3 \to 0$ [3,6]. We seek a numerical solution to Eq.(1) using exponential finite-difference method.

2.1. Exponential Finite-Difference:

The exponential finite-difference method that we applied to solve Eq.(1) was originally developed by Bhattachary [7] and used to solve one dimensional heat conduction in a solid slab [8,9]. It is also used to solve the Korteweg-de Vries equation [2].

We assume that F(u) denote to any continuous differential function. Multiplying Eq.(1) by the derivative of F lead to the following equation:

$$\frac{\partial F}{\partial u}\frac{\partial u}{\partial t}=F'(u).(-m_1 u\frac{\partial u}{\partial x}-m_2\frac{\partial^2 u}{\partial x^2}-m_3\frac{\partial^3 u}{\partial x^3})$$

and

$$\frac{\partial F}{\partial t}=F'(u).(-m_1 u\frac{\partial u}{\partial x}-m_2\frac{\partial^2 u}{\partial x^2}-m_3\frac{\partial^3 u}{\partial x^3}) \qquad (3)$$

Using the usual forward difference replacement to $\frac{\partial F}{\partial t}$ we obtain the finite-difference representation of Eq. (3) as:

$$u_i^{j+1}=F(u_i^j)+k\,F(u_i^j)\left[-m_1 u_i^j(\frac{\partial u}{\partial x})_i^j-m_2(\frac{\partial^2 u}{\partial x^2})_i^j-m_3(\frac{\partial^3 u}{\partial x^3})_i^j\right]$$

Where k is the time step. Now, if we let $F(u)=\ln u$ then we obtained the exponential finite-difference scheme as:

$$u_i^{j+1}=u_i^j.\exp\left\{\frac{k}{u_i^j}\left[-m_1 u_i^j(\frac{\partial u}{\partial x})_i^j-m_2(\frac{\partial^2 u}{\partial x^2})_i^j-m_3(\frac{\partial^3 u}{\partial x^3})_i^j\right]\right\} \qquad (4)$$

The finite-difference for the derivatives have been taken following form:[5]

$$(\frac{\partial u}{\partial x})_i^j=\frac{-3u_i^{j+1}+4u_{i+1}^{j+1}-u_{i+2}^{j+1}}{2h} \qquad (5)$$

$$(\frac{\partial^2 u}{\partial x^2})_i^j=\frac{2u_i^{j+1}-5u_{i+1}^{j+1}+4u_{i+2}^{j+1}-u_{i+3}^{j+1}}{h^2} \qquad (6)$$

$$(\frac{\partial^3 u}{\partial x^3})_i^j=\frac{-5u_i^{j+1}+18u_{i+1}^{j+1}-24u_{i+2}^{j+1}+14u_{i+3}^{j+1}-3u_{i+4}^{j+1}}{2h^3};\ i=1,2 \qquad (7)$$

$$(\frac{\partial u}{\partial x})_i^j=\frac{u_{i+1}^{j+1}-u_{i-1}^{j+1}}{2h} \qquad (8)$$

$$(\frac{\partial^2 u}{\partial x^2})_i^j=\frac{u_{i+1}^{j+1}-2u_i^{j+1}+u_{i-1}^{j+1}}{h^2} \qquad (9)$$

$$(\frac{\partial^3 u}{\partial x^3})_i^j=\frac{u_{i+2}^{j+1}-2u_{i+1}^{j+1}+2u_{i-1}^{j+1}-u_{i-2}^{j+1}}{2h^3};\ i=3...N-2 \qquad (10)$$

$$(\frac{\partial u}{\partial x})_i^j=\frac{3u_i^{j+1}-4u_{i-1}^{j+1}+u_{i-2}^{j+1}}{2h} \qquad (11)$$

$$(\frac{\partial^2 u}{\partial x^2})_i^j=\frac{2u_i^{j+1}-5u_{i-1}^{j+1}+4u_{i-2}^{j+1}-u_{i-3}^{j+1}}{h^2} \qquad (12)$$

$$(\frac{\partial^3 u}{\partial x^3})_i^j=\frac{5u_i^{j+1}-18u_{i-1}^{j+1}+24u_{i-2}^{j+1}-14u_{i-3}^{j+1}+3u_{i-4}^{j+1}}{2h^3};\ i=N-1,N \qquad (13)$$

2.2. Classical Explicit Finite-Difference Method:

The classical explicit finite-difference method (CEFD) can be applied to Eq. (1) and we have the form:

$$(\frac{\partial u}{\partial t})_i^{j+1}+m_1 u_i^j(\frac{\partial u}{\partial x})_i^j+m_2(\frac{\partial^2 u}{\partial x^2})_i^j+m_3(\frac{\partial^3 u}{\partial x^3})_i^j=0 \qquad (14)$$

then

$$u_i^{j+1}=u_i^j+k(-m_1 u_i^j(\frac{\partial u}{\partial x})_i^j-m_2(\frac{\partial^2 u}{\partial x^2})_i^j-m_3(\frac{\partial^3 u}{\partial x^3})_i^j) \qquad (15)$$

Now , we use the equations (5)-(13) in Eq. (15) we obtained the classical explicit finite-difference equations.

3. Analytical Solution

Let us take the initial value u(x, 0) as

$$u(x,0)=\frac{6.m_2^2}{25.m_1 m_3}\left[1+\tanh\left(\frac{m_2 x}{10.m_3}\right)+\frac{1}{2}\sec h^2\left(\frac{m_2 x}{10.m_3}\right)\right]$$

The analytical solution of this problem with the above initial data is: (see [3])

$$u(x,t)=\frac{6m_2^2}{25m_1 m_3}\left[1+\tanh\left(\frac{m_2}{10m_3}\left(x-\frac{6m_2^2 t}{25m_3}\right)\right)+\frac{1}{2}\sec h^2\left(\frac{m_2}{10m_3}\left(x-\frac{6m_2^2 t}{25m_3}\right)\right)\right] \qquad (16)$$

4. Numerical Results and Discussion:

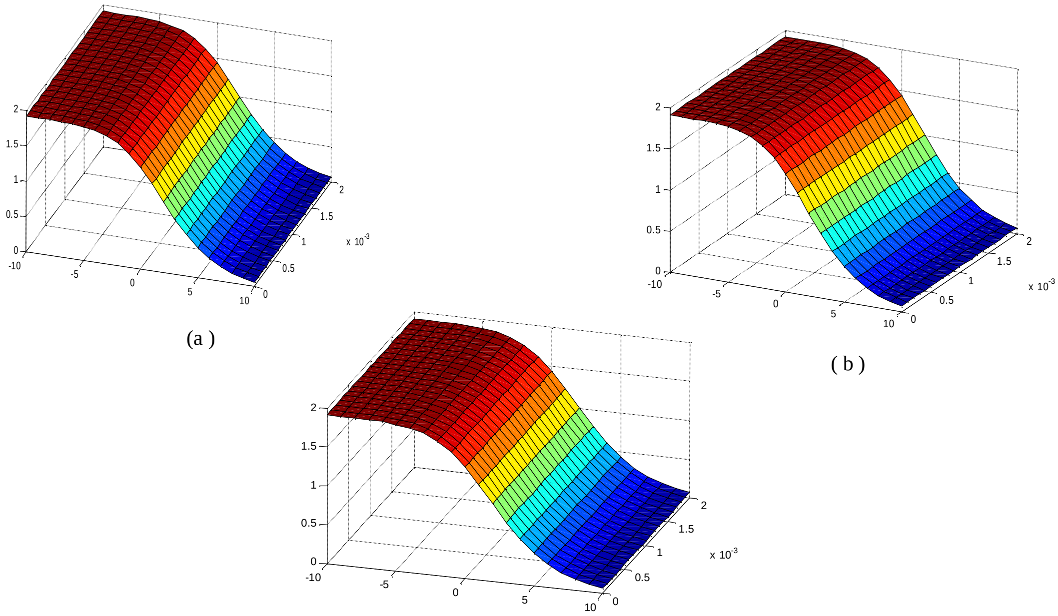
In this section, we present numerical results and compare those with analytical results. We apply the exponential finite-difference and the classical explicit finite-difference(CEFD) schemes to nonlinear KdV-B equation to compute solutions numerically and the compare these solutions with exact solutions at various times. All numerical computations we performed with the space step h=1,2 and the time step k=0.0001, and we take the parameters $m_1=1, m_2=-2, and\ m_3=1$. The results obtained for problem is displayed in Tables 1-2 for times t=0.002 and t=0.0001 respectively. According to the results presented here, the exponential finite-difference scheme behaves better than the other numerical scheme at small times. We see an excellent agreement of the numerical results with the analytical results.

Table (1): Comparison Numerical and Exact solution of the KdV-B equation at t=0.002

x	EXPONENTIAL	CEFD	EXACT
-10	1.919379002634842	1.919379191388729	1.919379808994134
-9	1.918643308672068	1.918644633876643	1.918643821094912
-8	1.917060591127396	1.917065926974663	1.917059152117866
-7	1.913722954398500	1.913722949715588	1.913699891657232
-6	1.906781239643815	1.906781242136646	1.906736712647412
-5	1.892835885409647	1.892835885224194	1.892754960258415
-4	1.866026301491624	1.866026300933741	1.865890978899002
-3	1.817448766909415	1.817448765469159	1.817246288809509
-2	1.735914795095435	1.735914792087203	1.735653118901616
-1	1.611343538917609	1.611343534006028	1.611065205290690
0	1.440592304868509	1.440592298776634	1.440368569203008
1	1.232347317115225	1.232347311477042	1.232244495952924
2	1.006349957875933	1.006349954010265	1.006390892416720
3	0.786237589842130	0.786237587888046	0.786392981751948
4	0.591000668951221	0.591000668241789	0.591214187873221
5	0.430510400722553	0.430510400558712	0.430730090382934
6	0.306113481426435	0.306113481413995	0.306307415711977
7	0.213810056492372	0.213810056488724	0.213965875257439
8	0.147446152916699	0.147446152912332	0.147557854272728
9	0.100767547648177	0.100767547637227	0.100848594270708
10	0.068439291585236	0.068439291569168	0.068497090192202

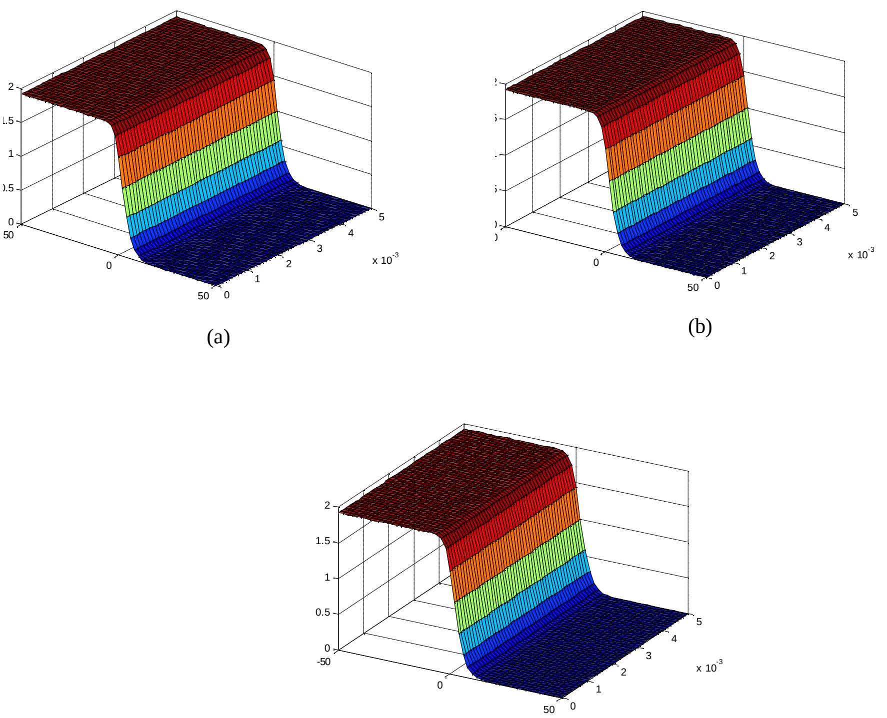
Table (2): Comparison Numerical and Exact solution of the KdV-B equation at t=0.0001

x	EXPONENTIAL	CEFD	EXACT
-50	1.920000000000000	1.920000000000000	1.920000000000000
-48	1.920000000000000	1.920000000000000	1.920000000000000
-46	1.920000000000000	1.920000000000000	1.920000000000000
-44	1.919999999999999	1.919999999999999	1.919999999999999
-42	1.919999999999995	1.919999999999995	1.919999999999995
-40	1.919999999999976	1.919999999999976	1.919999999999976
-38	1.919999999999881	1.919999999999881	1.919999999999880
-36	1.919999999999408	1.919999999999408	1.919999999999404
-34	1.919999999997066	1.919999999997066	1.919999999997046
-32	1.919999999985468	1.919999999985468	1.919999999985367
-30	1.919999999928025	1.919999999928025	1.919999999927524
-28	1.919999999643509	1.919999999643509	1.919999999641029
-26	1.919999998234346	1.919999998234346	1.919999998222064
-24	1.919999991255290	1.919999991255290	1.919999991194480
-22	1.919999956694102	1.919999956694102	1.919999956393214
-20	1.919999785580899	1.919999785580898	1.919999784093933
-18	1.919998938816973	1.919998938816973	1.919998931488683
-16	1.919994753189718	1.919994753189717	1.919994717292689
-14	1.919974114126656	1.919974114126648	1.919973940636680
-12	1.919872899009072	1.919872899008896	1.919872084813836
-10	1.919382507871398	1.919382507867956	1.919378919645133
-8	1.917068927584297	1.917068927532305	1.917055025946910
-6	1.906761136156680	1.906761135669220	1.906718956094980
-4	1.865909786484020	1.865909784408552	1.865825241803903
-2	1.735555298273523	1.735555295524832	1.735467399791167
0	1.440034892014480	1.440034891591760	1.440018431823051
2	1.005932997922627	1.005932997668404	1.005977376943306
4	0.590866411570610	0.590866411551361	0.590888328292627
6	0.306131865004398	0.306131863466041	0.306111484638387
.	.	.	.
.	.	.	.
.	.	.	.
.	.	.	.
44	0.000000087286934	0.000000087286925	0.000000087249913
46	0.000000039154376	0.000000039154347	0.000000039203914
48	0.000000017593195	0.000000017593182	0.000000017615454
50	0.000000007905132	0.000000007905126	0.000000007915134



(a) (b) (c)

Fig.(1): The numerical results for the KdV-B equation at h=1 by using: (a) Exponential finite-difference method; (b) Classical explicit finite-difference method; (c) The analytical solution



(a) (b) (c)

Fig. (2):The numerical results for the KdV-B equation at h=2 by using:(a) Exponential finite-difference method;(b)Classical explicit finite-difference method;(c)The analytical solution

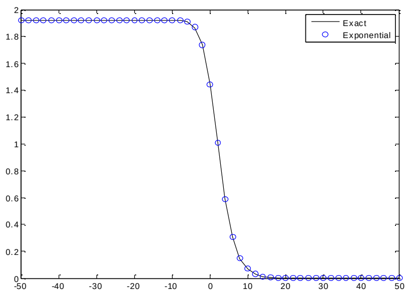


Fig.(3): Comparison of exponential finite-difference and exact solutions at t=0.003

Fig.1 and 2 show that the numerical solutions of the two methods (exponential and classical finite-difference) and the analytical solution also satisfy the physical behavior of the KdV-B equation. Comparison of the results obtained by the exponential finite-difference and exact solution is illustrated in Fig 3 for time t=0.003

4. Conclusion

The exponential finite-difference method is applied to solve the KdV-B equation. The technique exhibits higher accuracy than the classical explicit finite-difference method with which it is compared with the exact solution. Therefore, it is concluded that the exponential finite-difference method can be used to produce reasonably accurate numerical solutions of the KdV-B equation at small times. And the main advantage of the method is that it can be applied directly to all types of partial differential equations without any need for restrictive assumptions on the boundary conditions.

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