Unconditionally Stable Crank-Nicolson Perfectly Matched Layer for Truncating Finite Difference Time domain Grids

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ABSTRACT

Efficient and unconditional stable Perfectly Matched Layer (PML) formulations are presented for truncating finite difference time domain grids. The formulations are based on incorporating the Crank-Nicolson scheme into the stretched coordinate PML approach. In these formulations, the field components are updated in a single step per time iteration. Numerical example carried out in one-dimensional domain shows that the formulations are both accurate and unconditionally stable.

Key words: Crank-Nicolson (CN), Stretched Coordinate Perfectly Matched Layer (PML), Finite Difference Time Domain (FDTD).

1. Introduction

In recent years, the Finite-Difference Time-Domain method (FDTD) [1] has been shown to be one of the most widely used numerical time-domain techniques in electromagnetism, as it covers many applications, such as antennas, optics, high-speed electronic circuits, and semiconductors, etc. Unfortunately, the stability of the FDTD method is limited by the Courant Friedrichs Lewy (CFL) stability condition. Since the CFL is determined by the smallest cell size in the domain, the FDTD analysis of fine geometric structures requires large number of time iterations. Hence, the elimination of the CFL stability limit is one of the latest challenges in the FDTD research. Very recently, an implicit scheme, known as the alternating direction implicit finite difference time domain (ADI-FDTD) method [2], has been used for solving electromagnetic problems where very fine meshes with respect to the wavelength is needed. This method is unconditionally stable and removes completely the CFL stability condition required by the conventional FDTD approach. On the other hand, the ADI-FDTD method is found to be less accurate than the conventional FDTD scheme, especially for large time step due to the increase in the numerical dispersion error of the ADI-FDTD formulations [3].

Another important issue in the FDTD research is the development of accurate Absorbing Boundary Conditions (ABCs) to truncate open region problems. Berenger's Perfectly Matched Layer (PML) [4, 5] has been shown to be one of the most effective FDTD ABCs. In recent years, different unconditional stable ADI implementations of the PML (ADI-PML) have been introduced [6-9]. Similar to the ADI-FDTD method, the accuracy of the ADI-PML formulations decreases with the increase of the time step.

In this paper, accurate and unconditionally stable formulations of the PML are presented. The formulations are based on incorporating the Crank Nicolson (CN) scheme into the PML formulations. These formulations, named as CN-PML, allow updating the field components from the n to n+1 time step in a single iteration rather that the two subiterations needed in the ADI-FDTD method. One dimensional numerical example has been carried out to validate the proposed formulations.

The paper is organized as follows. In section II, the formulations of CN-PML

scheme are presented. Section III includes the results of a numerical test to show the effectiveness of the proposed method. Finally, a summary and conclusions are included in section IV.

2. Formulation

Using the stretched coordinate PML formulation [5], the field equations for an x-directed, z-polarized transverse electromagnetic (TEM) wave propagating in one-dimensional source free homogenous, isotropic, and lossy medium can be written in the frequency domain as

$$j\omega\varepsilon_r E_z + \frac{\sigma}{\varepsilon_0}E_z = c\frac{1}{S_x}\frac{\partial}{\partial x}H_y \tag{1}$$

$$j\omega H_y = c\frac{1}{S_x}\frac{\partial}{\partial x}E_z \tag{2}$$

where c is the speed of light in vacuum, ε_r is the permittivity, σ is the conductivity of the medium, and S_x is the PML stretched coordinate variable defined [5]

$$\frac{1}{S_x} = \frac{j\omega}{j\omega+\sigma_x/\varepsilon_0} = 1-\frac{\sigma_x/\varepsilon_0}{j\omega+\sigma_x/\varepsilon_0} \tag{3}$$

where σ_x is the conductivity profile along the x-direction in the PML region. Using (3) and noting that $j\omega \to \partial/\partial t$, (1) and (2) can be written in the time domain as

$$\varepsilon_r\frac{\partial}{\partial t}E_z + \frac{\sigma}{\varepsilon_0}E = c\frac{\partial}{\partial x}H_y - f_{zx} \tag{4}$$

$$\frac{\partial}{\partial t}H_y = c\frac{\partial}{\partial x}E_z - g_{yx} \tag{5}$$

where f_{zx} and g_{yx} are given by

$$\frac{\partial}{\partial t}f_{zx} + \frac{\sigma_x}{\varepsilon_0}f_{zx} = c\frac{\sigma_x}{\varepsilon_0}\frac{\partial}{\partial x}H_y \tag{6}$$

$$\frac{\partial}{\partial t}g_{yx} + \frac{\sigma_x}{\varepsilon_0}g_{yx} = c\frac{\sigma_x}{\varepsilon_0}\frac{\partial}{\partial x}E_z \tag{7}$$

Using the Crank-Nicolson scheme, (4)-(7) can be discretized as

$$\begin{aligned}E_{z_i}^{n+1} &= G_a E_{z_i}^n - \frac{\Delta t}{2}(f_{zx_i}^{n+1} + f_{zx_i}^n)\\ &+\chi G_b(H_{y_{i+1/2}}^{n+1} - H_{y_{i-1/2}}^{n+1} + H_{y_{i+1/2}}^n - H_{y_{i-1/2}}^n)\end{aligned} \tag{8}$$

$$\begin{aligned}H_{y_{i+1/2}}^{n+1} &= H_{y_{i+1/2}}^n - \frac{\Delta t}{2}(g_{yx_{i+1/2}}^{n+1} + g_{yx_{i+1/2}}^n)\\ &+\chi(E_{z_{i+1}}^{n+1} - E_{z_i}^{n+1} + E_{z_{i+1}}^n - E_{z_i}^n)\end{aligned} \tag{9}$$

$$\begin{aligned}f_{zx_i}^{n+1} &= r_{0_i} f_{zx_i}^n\\ &+ r_{1_i}\frac{c}{\Delta x}(H_{y_{i+1/2}}^{n+1} - H_{y_{i-1/2}}^{n+1} + H_{y_{i+1/2}}^n - H_{y_{i-1/2}}^n)\end{aligned} \tag{10}$$

$$\begin{aligned}g_{yx_{i+1/2}}^{n+1} &= r_{0_{i+1/2}} g_{yx_{i+1/2}}^n\\ &+ r_{1_{i+1/2}}\frac{c}{\Delta x}(E_{z_{i+1}}^{n+1} - E_{z_i}^{n+1} + E_{z_{i+1}}^n - E_{z_i}^n)\end{aligned} \tag{11}$$

where Δx is the space cell size, Δt is the time step, $\chi = c\Delta t/\Delta x$, and

$$\begin{aligned}&G_a = \frac{\varepsilon_r - p}{\varepsilon_r + p},\ \text{and}\ G_b = \frac{1}{\varepsilon_r + p}\\ &r_{0_i} = \frac{1-q_i}{1+q_i},\ \text{and}\ r_{1_i} = \frac{q_i}{1+q_i}\end{aligned} \tag{12}$$

with

$$p = \frac{\Delta t p}{2\varepsilon_0}\ \text{and}\ q_i = \frac{\Delta t\sigma_{x_i}}{2\varepsilon_0} \tag{13}$$

From (8), it can be seen that E_z^{n+1} can not be updated directly as it depends on H_y^{n+1}. Therefore, by substituting (9), (10) and (11) into (8), and after some manipulations, an implicit update for E_z^{n+1} can be obtained from

$$\begin{aligned}&-A_i^- E_{z_{i-1}}^{n+1} + A_i E_{z_i}^{n+1} - A_i^+ E_{z_{i+1}}^{n+1} = G_a E_{z_i}^n\\ &-G_b\frac{\Delta t}{2}(1+r_{0_i})f_{zx_i}^n + 2G_a(1-r_{1_i})\chi\left(H_{y_{i+1/2}}^n - H_{y_{i-1/2}}^n\right)\\ &-G_b(1-r_{1_i})\chi\frac{\Delta t}{2}\left[(1+r_{0_{i+1/2}})g_{zx_{i+1/2}}^n - (1+r_{0_{i-1/2}})g_{zx_{i-1/2}}^n\right]\\ &+A_i^- E_{z_{i-1}}^n - A_i E_{z_i}^n + A_i^+ E_{z_{i+1}}^n\end{aligned} \tag{14}$$

where

$$A_i^{\pm} = \Lambda(1-r_{1_{i\pm1/2}})\ \text{and}\ A_i = 1+\Lambda(2-r_{1_{i+1/2}} - r_{1_{i-1/2}}) \tag{15}$$

with

$$\Lambda = \chi^2 G_b(1-r_{1_i}) \tag{16}$$

It is clear that the left hand side of (14) forms a tri-diagonal matrix and the corresponding linear system of equations can be solved for E_z^{n+1} easily [2]. Once E_z^{n+1} is obtained, g_{yx}^{n+1}, H_y^{n+1}, and f_{zx}^{n+1} can be updated explicitly from (11), (9) and (10), respectively. It is important to note that the above formulations are applied in the PML regions at the domain boundaries. In non-PML regions, it is only required to set the coefficients r_0 and r_1 defined in (12) to unity and zero, respectively. Finally, it must be mentioned that the extension of the above formulations to the two or the three dimensional problems necessitate the solution of a linear system of equations at

each time step. To reduce the resultant computational time and storage requirements, techniques similar to those reported in [10, 11] can be used.

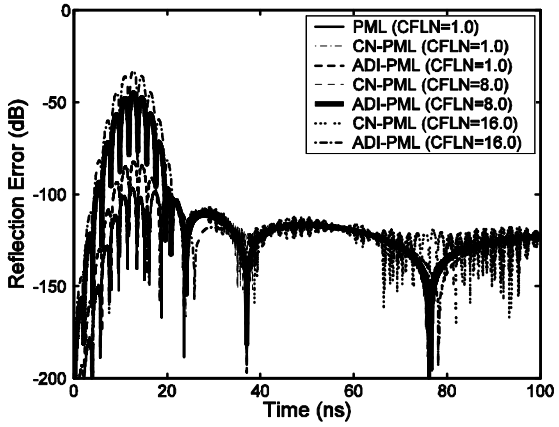

Figure 1. Reflection error for the proposed CN-PML and for ADI-PML formulations as compared with the conventional PML reflections as observed one cell from the PML/computational domain interface.

4. Simulation Study

To show the validity of the proposed formulations, a numerical example carried out in a one dimensional domain is presented. A point source, with E_z polarization, was excited at the centre of 100 cells isotropic, homogeneous, and lossy computational domain with the parameters of ε_r=2, and σ=0.1S/m. The computational domain extends in the x direction and discretized with a space cell size of Δx =1.0mm. The excitation was a Gaussian pulse with a maximum frequency content of 500MHz. Both ends of the computational domain were terminated by eight additional PML layers with a quadratic conductivity profile and with 0.001% theoretical reflection coefficient, i.e., PML[8, 2, 0.001%], as defined in [4]. Figure 1 shows the reflection error of the PML for the CN-PML and for the ADI-PML formulations [9] as obtained with different CFL number (CFLN) defined as CFLN= $\Delta t / \Delta t_{max}^{FDTD}$, where Δt_{max}^{FDTD} is the maximum stability limit of the conventional FDTD algorithm, in this test = Δt_{max}^{FDTD} =4.7ps. The results obtained using the conventional PML implementation of (4)-(7) with CFLN=1.0 are also shown in Fig. 1. The reflection error was computed one space cell away from the PML/computational domain interface as

$$R_{dB}(f) = 20\log_{10}\left(\frac{\left|E_z^R(t) - E_z^T(t)\right|}{\max\left[\left|E_z^R(t)\right|\right]}\right) \qquad (17)$$

where $E_z^T(t)$ is the electric field computed using the test domain and $E_z^R(t)$ is the reference electric field computed for each CFLN value by using a larger domain with the size of 1000 and truncated by 32 PML layers with the parameters of PML[32, 4, 0.001%]. As can be seen from Fig. 1, the reflection error of the ADI-PML formulations increases as the CFLN values increases. On the other hand, this is not observed for the CN-PML, where the reflection error maintains the same level as the conventional PML results.

5. Conclusion

In this paper, unconditional stable PML formulations, based on the CN scheme, are presented for truncating FDTD domains. The proposed formulations allow updating the field components only in a single iteration per time step. Numerical example carried out in a one dimensional domain shows that the formulations remain stable beyond the stability limit of the conventional FDTD method. Hence, the simulation time can be reduced by increasing the time step without affecting the level of accuracy. The formulations can be extended to the two and the three dimensions in a similar manner. In these cases, the solution of a linear system of equations at each time step is needed, which will result in an increase in the computational time and storage. Techniques similar to those reported in [10, 11] can be used to reduce the resultant computational time and storage requirements.

References:

[1] A. Taflove and S.C. Hangess, Computational electrodynamics: The

Finite-Difference Time-Domain Method, 2nd ed. Boston, Artech-House, 2000.
[2] T. Namiki, ``A new FDTD algorithm based on alternating-direction implicit method," IEEE Trans. Microwave Theory Tech., Vol. 47, No. 10, 1999, pp. 2003-2007.
[3] S.G. Carcia, T.W. Lee, and S.C. Hagness, ``On the accuracy of the ADI-FDTD method," IEEE Antennas and Wireless Propagation Letters Vol. 1, 2002, pp. 31--34.
[4] J.-P. Berenger, ``A perfectly matched layer for the absorption of electromagnetic waves," J. Comput. Phys., Vol. 114, 1994, pp. 185-200.
[5] W.C. Chew, and W.H.~Weedon, ``A 3-D perfectly matched medium from modified Maxwell's equation with stretched coordinates," Microwave Opt. Technol. Lett., Vo. 7, No. 13, 1994, pp. 599-604.
[6] S. D. Gedney, G. Liu, J. A. Roden, and A. Zhu, ``Perfectly matched layer media with CFS for an unconditionally stable ADI-FDTD method," IEEE Trans. Antennas Propagat., Vol. 49, No. 11, 2001, pp. 1554-1559.
[7] A. P. Zhao, ``Uniaxial perfectly matched layer media for an unconditionally stable 3-D ADI-FD-TD method," IEEE Microwave Wireless Compon. Lett., Vol. 12, No. 12, 2002, pp. 497-499.
[8] S. Schmidt, and G. Lazzi, ``Extension and validation of a perfectly matched layer formulation for the unconditionally stable D-H FDTD method," IEEE Microwave Wireless Compon. Lett., Vol. 13, No. 8, 2003, pp. 345-347.
[9] O. Ramadan, ``Unconditionally stable ADI-FDTD implementation of PML for frequency dispersive debye media," Electron. Lett., Vol. 40, No. 4, 2004, pp. 230-232.
[10] A.P. Zhao, "A novel implementation for two-dimensional unconditionally stable FDTD method", Microwave and Optical Technology Letters, Vol. 38, 2003, pp. 457-462.
[11] G. Sun, and C.W. Trueman, "Approximate Crank-Nicolson schemes for the 2-D finite-difference time-domain method for TEz waves", IEEE Transactions on Antennas and Propagation, Vol. 52, 2004, pp. 2963-2972.