BINOMIAL AND TRINOMIAL TREES VERSUS BJERKSUND AND STENSLAND APPROXIMATIONS FOR AMERICAN OPTIONS PRICING

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Abstract

Numerical methods like binomial and trinomial trees and finite difference methods can be used to price a wide range of options contracts for which there are no known analytical solutions. American options are the most famous of that kind of options. Besides numerical methods, American options can be valued with the approximation formulas. The authors of the most famous approximation formulas are Bjerksund and Stensland ([3] and [4]).

Comparative analysis of numerical methods for American option pricing and Bjerksund and Stensland formulas for approximation values of American options is carried out in this paper.

When the value of American option is approximated by Bjerksund-Stensland analytical formulas, the computer time spent to carry out that calculation is very short (it can be considered as instantaneous). The computer time spent using numerical methods can vary from less than one second to several minutes or even hours. It is clear that more often increasing the time of computer calculation greater precision is obtained. However to be able to conduct a comparative analysis of numerical methods (binomial trees, trinomial trees) and Bjerksund and Stensland formulas for approximation values of American options, we will limit computer calculation time of numerical method to less than one second, which nearly corresponds to the calculation time of the Bjerksund and Stensland approximation. Therefore, we ask the question: Which method will be most accurate at nearly the same computer calculation time?

Keywords – *option pricing, binomial and trinomial trees,* Bjerksund and Stensland formulas

1 INTRODUCTION

The American option can be exercised at any time up to its expiration date. This added freedom complicates the valuation of American options relative to their European counterparts. With a few exceptions, it is not possible to find an exact formula for the value of American options. Several researchers have, however, come up with excellent closed-form approximations (Barone-Adesi, G. and R. E. Whaley [1], Bjerksund, P. and G. Stensland [3],[4]. These approximations have become especially popular because they execute more quickly on computers than the numerical techniques.

Numerical methods that can be used for evaluation of American options are binomial and trinomial trees and finite difference methods. These methods are more flexible then analytical solutions and can be used to price a wide range of options contracts for which there are no known analytical solutions including the American options.

The binomial method was first published by Cox, Ross and Rubinstein [6] and Rendleman and Bartter [9]. Trinomial trees were introduced in option pricing by Boyle [5] and are similar to binomial trees. The main objection to these methods is that the computing time required for their algorithms is longer than for the analytical expressions. But with the development of computer technology computers become faster and the computation time is reduced significantly. The question arises of whether the price of American options obtained by numerical methods in a short time (less than one second) is closer to the correct value of the option than the price obtained by an approximation formula. This paper will try to give answers to this question by evaluating 280 American options by binomial and trinomial trees and Bjerksund and Stensland formulas for approximation values of American options.

The paper is organized as follows: following this introduction, in Section 2, we describe the binomial and trinomial model for valuing options. In Section 3 we describe Bjerksund and Stensland formulas for approximation values of American options. In Section 4 we conduct a comparative analysis of specified numerical methods and approximation formulas. Section 5 summarizes the paper and indicates the possible directions for further research.

2 BINOMIAL AND TRINOMIAL MODEL FOR VALUING OPTIONS

2.1 Binomial model

The procedure followed by binomial model is to assume that the stock price follows a discrete time process. The life of the option $T - t$ is decomposed into n equal time steps of length $(\Delta t = (T - t)/n)$. At each time interval ($t_j = j \cdot \Delta t, \quad j = 0, 1, ..., n$), it is assumed that the underlying instrument will move up or down by a specific factor (u or d where, by definition $u \geq 1$, and $0 < d \leq 1$) per step of the tree with probability p, 1-p respectively . So, if S is the current price, then in the next period the price will either be $S_{up} = S \cdot u$ or $S_{down} = S \cdot d$. The binomial tree of stock's price is best illustrated in a Fig. 1.

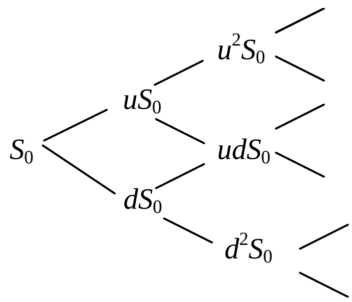

Fig. 1. Binomial tree

The up and down jump factors and corresponding probabilities are chosen to match the first two moments of the stock price distribution (mean and variance). There are, however, more unknowns than there are equations in this set of restrictions, implying that there are many ways of choosing the parameters and still satisfy the moment restrictions. Cox, Ross and Rubinstein [6] set the up and down parameters to

$$u = e^{\sigma\sqrt{\Delta t}}, \; d = e^{-\sigma\sqrt{\Delta t}}, \tag{1}$$

where σ is volatility of the relative price change of the underlying stock price. The probability of the stock price increasing at the next time step is:

$$p = \frac{e^{r\Delta t} - d}{u - d}, \tag{2}$$

where r is risk-free interest rate.

At each final node of the tree i.e. at expiration of the option the option value is simply its intrinsic, or exercise, value

Max [($S_n - K$), 0], for a call option

Max [($K - S_n$), 0], for a put option,

where K is the strike price and S_n is the spot price of the underlying asset at the n^{th} period.

Once the above step is complete, the option value is then found for each node, starting at the penultimate time step, and working back to the first node of the tree (the valuation date) where the calculated result is the value of the option.

Under the risk neutrality assumption, today's fair price of a derivative is equal to the expected value of its future payoff discounted by the risk free rate. Therefore, expected value is calculated using the option values from the later two nodes (Option up and Option down) weighted by their respective probabilities (probability p of an up move in the underlying, and probability 1-p of a down move). The expected value is then discounted at r, the risk free rate corresponding to the life of the option.

The following formula to compute the expectation value is applied at each node:

$$C_{t-\Delta t,i} = e^{-r\Delta t}\left(pC_{t,i+1} + (1-p)C_{t,i}\right) \tag{3}$$

where

$C_{t,i}$ is the option's value for the i^{th} node at time t.

This result is the “Binomial Value”. It represents the fair price of the derivative at a particular point in time (i.e. at each node), given the evolution in the price of the underlying asset to that point. It is the value of the option if it were to be held—as opposed to exercised at that point.

For an American option, since the option may either be held or exercised prior to expiry, the value at each node is: Max (Binomial Value, Exercise Value). The value of the initial node presents the required fair price of the option.

2.2 Trinomial model

Under the trinomial model, in each period, the prices can go up, down or remain unchanged. The term "lattice" implies two or more branches protruding from the node of a tree. In the case of a binomial lattice there are two branches, three in the case of a trinomial, and so on. Where there are more than two branches, the lattice can be called a multinomial lattice.

A trinomial lattice works on the same principles as the binomial lattice, but assumes that the prices may also remain constant. So in the first step, the prices may go up, down or remain unchanged. For each of the three outcomes, there will be three outcomes each in the second time step, but the second outcome of the first node in the second step will be the same as the first outcome of the second node in the second step and so on.

The expected results are attained much faster, as the branches become intractable at a much earlier period of time. Trinomial trees can be used as an alternative to binomial trees, where there are numerous time steps. It is to be noted that the trinomial tree computation procedure is exactly the same as for the binomial model.

As the name suggests, trinomial model uses a similar approach to be binomial one. But the hedging and replication arguments do not take place in constructing trinomial trees. For a non - dividend paying stock, parameter values that match the mean and standard deviation of price changes are given below:

$$u = e^{\sigma\sqrt{3\Delta t}}, d = 1/u \,, \tag{4}$$

$$p_u = \sqrt{\frac{\Delta t}{12\sigma^2}}\left(r - \frac{1}{2}\sigma^2\right) + \frac{1}{6}, \tag{5}$$

$$p_d = -\sqrt{\frac{\Delta t}{12\sigma^2}}\left(r - \frac{1}{2}\sigma^2\right) + \frac{1}{6}, \tag{6}$$

$$p_m = 1 - p_u - p_d \,, \tag{7}$$

where u, d and r have the same meaning as in binomial model, σ is stock volatility, while p_u, p_d and p_m denote probabilities of the price going up, down or remaining unchanged, respectively.

Once the tree of prices has been calculated, the option price is found at each node largely as for the binomial model, by working backwards from the final nodes to today. The difference being that the option value at each non-final node is determined based on the three (as opposed to two) later nodes and their corresponding probabilities.

3 THE BJERKSUND AND STENSLAND (1993) AND (2002) APPROXIMATION

The Bjerksund and Stensland [3] approximation can be used to price American options on stocks, futures and currencies. Bjerksund and Stensland's approximation is based on an exercise strategy corresponding to a flat boundary I (trigger price).

Given this feasible but non-optimal strategy, the American call boils down to: (i) a European up-and-out call with knock-out barrier I, strike K, and maturity date T; and (ii) a rebate I-K that is received at the knock-out date if the option is knocked out prior to the maturity date.

Their American call approximation is

$$\begin{aligned} c = \alpha S^{\beta} - \alpha\phi(S,T,\beta,I,I) + \phi(S,T,1,I,I) - \phi(S,T,1,K,I) - \\ - K\phi(S,T,0,I,I) + K\phi(S,T,0,K,I), \end{aligned} \tag{8}$$

where

$$\alpha = (I-K)I^{-\beta},\ \beta = \left(\frac{1}{2} - \frac{b}{\sigma^2}\right) + \sqrt{\left(\frac{b}{\sigma^2} - \frac{1}{2}\right)^2 + 2\frac{r}{\sigma^2}}. \tag{9}$$

The function $\phi(S,T,\gamma,H,I)$ is given by:

$$\phi(S,T,\gamma,H,I) = e^{\lambda}S^{\gamma}\left[N(d) - \left(\frac{I}{S}\right)^{\kappa} N\left(d - \frac{2\ln\left(\frac{I}{S}\right)}{\sigma\sqrt{T}}\right)\right], \tag{10}$$

$$\lambda = \left[-r + \gamma b + \frac{1}{2}\gamma(\gamma-1)\sigma^2\right]T,\ d = \frac{\ln\left(\frac{S}{H}\right) + \left[b + \left(\gamma - \frac{1}{2}\right)\sigma^2\right]T}{\sigma\sqrt{T}},\ \kappa = \frac{2b}{\sigma^2} + (2\gamma - 1),$$

And the trigger price I is defined as

$$I = B_0 + (B_\infty - B_0)\left(1 - e^{h(T)}\right), \tag{11}$$

$$h(T) = -\left(bT + 2\sigma\sqrt{T}\right)\left(\frac{B_0}{B_\infty - B_0}\right), B_0 = \frac{\beta}{\beta-1}K,\ B_\infty = \max\left\{K, \frac{r}{r-b}K\right\}.$$

If $S>I$, it is optimal to exercise the option immediately and the value must be equal to the intrinsic value of S-X. On the other hand, if $b \geq r$, it will never be optimal to exercise the American call option before expiration, and the value can be found using Black-Scholes formula [2]. The value of the American put is given by Bjerksund and Stensland put-call transformation:

$$p(S,K,T,r,b,\sigma) = c(S,K,T,r-b,-b,\sigma). \tag{12}$$

The Bjerksund and Stensland [4] approximation divides the time to maturity into two parts, each with a separate flat exercise boundary. They extend the flat boundary approximation above by allowing for one flat boundary I_1 that is valid from date 0 to date t, and another flat boundary I_2 that is valid from date t to date T, where $0 < t < T$. Their American call approximation is:

$$\begin{aligned} c = \alpha_2 S^{\beta} - \alpha_2\phi(S,t_1,\beta,I_2,I_2) + \phi(S,t_1,1,I_2,I_2) - \phi(S,t_1,1,I_1,I_2) - \\ - K\phi(S,t_1,0,I_2,I_2) + K\phi(S,t_1,0,I_1,I_2) + \alpha_1\phi(S,t_1,\beta,I_1,I_2) - \\ - \alpha_1\Psi(S,T,\beta,I_1,I_2,I_1,t_1) + \Psi(S,T,1,I_1,I_2,I_1,t_1) - \Psi(S,T,1,K,I_2,I_1,t_1) - \\ - K\Psi(S,T,0,I_1,I_2,I_1,t_1) + \Psi(S,T,0,K,I_2,I_1,t_1), \end{aligned} \tag{13}$$

where:

$$\alpha_1 = (I_1 - K)I_1^{-\beta},\quad \alpha_2 = (I_2 - K)I_2^{-\beta}, \beta = \left(\frac{1}{2} - \frac{b}{\sigma^2}\right) + \sqrt{\left(\frac{b}{\sigma^2} - \frac{1}{2}\right)^2 + 2\frac{r}{\sigma^2}}.$$

The function $\phi(S,T,\gamma,H,I)$ is given by:

$$\phi(S,T,\gamma,H,I) = e^{\lambda}S^{\gamma}\left[N(-d) - \left(\frac{I}{S}\right)^{\kappa} N(-d_2)\right], \tag{14}$$

$$\lambda = -r + \gamma b + \frac{1}{2}\gamma(\gamma-1)\sigma^2,\ d = \frac{\ln\left(\frac{S}{H}\right) + \left[b + \left(\gamma - \frac{1}{2}\right)\sigma^2\right]T}{\sigma\sqrt{T}},$$

$$d_2 = \frac{\ln\left(\frac{I^2}{SH}\right) + \left[b + \left(\gamma - \frac{1}{2}\right)\sigma^2\right]T}{\sigma\sqrt{T}},\ \kappa = \frac{2b}{\sigma^2} + (2\gamma - 1),$$

The trigger price I is defined as:

$$\begin{aligned} I_1 &= B_0 + (B_\infty - B_0)\left(1 - e^{h_1}\right), \\ I_2 &= B_0 + (B_\infty - B_0)\left(1 - e^{h_2}\right), \end{aligned} \tag{15}$$

$$h_1 = -\left(bt_1 + 2\sigma\sqrt{t_1}\right)\left(\frac{K^2}{(B_\infty - B_0)B_0}\right), h_2 = -\left(bT + 2\sigma\sqrt{T}\right)\left(\frac{K^2}{(B_\infty - B_0)B_0}\right),$$

$$t_1 = \frac{1}{2}\left(\sqrt{5} - 1\right)T\ ,\ B_0 = \frac{\beta}{\beta - 1}K\ ,\ B_\infty = \max\left\{K, \frac{r}{r-b}K\right\}.$$

Moreover, the function $\Psi(S,T,\gamma,H,I_2,I_1,t_1,r,b,\sigma)$ is given by:

$$\Psi(S,T,\gamma,H,I_2,I_1,t_1,r,b,\sigma) = e^{\lambda T}S^{\gamma}\left[M\left(-e_1, -f_1, \sqrt{\frac{t_1}{T}}\right) - \left(\frac{I_2}{S}\right)^{\kappa} M\left(-e_2, -f_2, \sqrt{\frac{t_1}{T}}\right) - \right.$$
$$\left. - \left(\frac{I_1}{S}\right)^{\kappa} M\left(-e_3, -f_3, \sqrt{\frac{t_1}{T}}\right) + \left(\frac{I_1}{I_2}\right)^{\kappa} M\left(-e_4, -f_4, \sqrt{\frac{t_1}{T}}\right)\right],$$

Where $M(\cdot,\cdot,\cdot)$ cumulative bivariate normal distribution and

$$e_1 = \frac{\ln\left(\frac{S}{I_1}\right) + \left[b + \left(\gamma - \frac{1}{2}\right)\sigma^2\right]t_1}{\sigma\sqrt{t_1}},\quad e_2 = \frac{\ln\left(\frac{I_2^2}{SI_1}\right) + \left[b + \left(\gamma - \frac{1}{2}\right)\sigma^2\right]t_1}{\sigma\sqrt{t_1}},$$

$$e_3 = \frac{\ln\left(\frac{S}{I_1}\right) - \left[b + \left(\gamma - \frac{1}{2}\right)\sigma^2\right]t_1}{\sigma\sqrt{t_1}},\quad e_4 = \frac{\ln\left(\frac{I_2^2}{SI_1}\right) - \left[b + \left(\gamma - \frac{1}{2}\right)\sigma^2\right]t_1}{\sigma\sqrt{t_1}},$$

$$f_1 = \frac{\ln\left(\frac{S}{H}\right) + \left[b + \left(\gamma - \frac{1}{2}\right)\sigma^2\right]T}{\sigma\sqrt{T}},\quad f_2 = \frac{\ln\left(\frac{I_2^2}{SH}\right) + \left[b + \left(\gamma - \frac{1}{2}\right)\sigma^2\right]T}{\sigma\sqrt{T}},$$

$$f_3 = \frac{\ln\left(\frac{I_1^2}{SH}\right) + \left[b + \left(\gamma - \frac{1}{2}\right)\sigma^2\right]T}{\sigma\sqrt{T}},\quad f_4 = \frac{\ln\left(\frac{SI_1^2}{HI_2^2}\right) + \left[b + \left(\gamma - \frac{1}{2}\right)\sigma^2\right]T}{\sigma\sqrt{T}}.$$

4 COMPARISON OF NUMERICAL METHODS AND BJERKSUND AND STENSLAND APPROXIMATIONS

Comparative analysis of observed models will be carried out by their application to pricing American put options on nondividend-paying stocks. We will compare the Bjerksund and Stensland (1993) and (2002) approximation with binomial model and trinomial model. We will limit computer calculation time of numerical method to less than one second, which nearly corresponds to the calculation time of the Bjerksund and Stensland approximation.

Since there is no formula that can calculate the exact value of American options offer, for the calculation of reference value, we will use trinomial model with a very large number of steps (5000 steps) that achieves high precision and the resulting value can be considered accurate. The calculation of the reference value using trinomial model in this analysis required over 100 hours of computer processing. The values obtained by the observed models are compared with the reference values. Errors of each particular model will be represented by the absolute value of the difference between the values obtained by the observed model and the reference value.

The survey is conducted by evaluating 280 American options with the exercise price of 150, and the volatility of 25%, with a risk-free interest rate of 6%. Time to maturity takes values of the interval $\left[0.05,1\right]$, and the current price of the option values are taken from the interval $\left[50,180\right]$.

The option values obtained by the analysis are given in Tables 1-5. In applying the binomial and trinominal model the biggest number (rounded to the tens) was taken for the number of periods, for which computer computation is less than one second.

The main aim is to find out whether the errors in the observed methods differ significantly.

For this purpose, we will apply the Friedman non-parametric test.

This test is used for more than two dependent variable samples measured using the sequence scale. The following hypotheses are set:

H_0 - there is no difference in the rank of model errors,

H_1 - there is a difference in the rank of model errors.

Figure 2 indicates the results of the conducted Friedman test. Friedman test was used to test the differences in the error ranks for all four models based on the results obtained for the option offer (picture 5.3). The obtained results show that in both cases there is a difference in ranks of error for the observed models, i.e. the initial hypothesis H_0 is rejected.

The binomial model has shown to be the best, followed by the trinomial, and Bjerksund Stenslandov-model (2002), with the Bjerksund Stenslandov-model (1993) taking the last position.

Table 1. Evaluating the American options from the sample using the trinominal model (n=5000)

		Asset price													
		50	60	70	80	90	100	110	120	130	140	150	160	170	180
	0,05	100,00000	90,00000	80,00000	70,00000	60,00000	50,00000	40,00000	30,00000	20,00000	10,19847	3,15076	0,47987	0,03346	0,00110
	0,1	100,00000	90,00000	80,00000	70,00000	60,00000	50,00000	40,00000	30,00000	20,00000	10,73557	4,34832	1,25706	0,25606	0,03739
T	0,15	100,00000	90,00000	80,00000	70,00000	60,00000	50,00000	40,00000	30,00000	20,00815	11,26361	5,22684	1,95552	0,58906	0,14462
i	0,2	100,00000	90,00000	80,00000	70,00000	60,00000	50,00000	40,00000	30,00000	20,07938	11,74576	5,94098	2,57269	0,95582	0,30776
m	0,25	100,00000	90,00000	80,00000	70,00000	60,00000	50,00000	40,00000	30,00000	20,19474	12,18433	6,55056	3,12455	1,32659	0,50549
e	0,3	100,00000	90,00000	80,00000	70,00000	60,00000	50,00000	40,00000	30,00000	20,33329	12,58522	7,08619	3,62410	1,68963	0,72317
	0,35	100,00000	90,00000	80,00000	70,00000	60,00000	50,00000	40,00000	30,00000	20,48377	12,95450	7,56603	4,08074	2,04012	0,95110
t	0,4	100,00000	90,00000	80,00000	70,00000	60,00000	50,00000	40,00000	30,00000	20,63969	13,29662	8,00188	4,50215	2,37668	1,18346
o	0,45	100,00000	90,00000	80,00000	70,00000	60,00000	50,00000	40,00000	30,00000	20,79738	13,61572	8,40195	4,89336	2,69902	1,41655
	0,5	100,00000	90,00000	80,00000	70,00000	60,00000	50,00000	40,00000	30,00000	20,95455	13,91454	8,77218	5,25892	3,00771	1,64802
m	0,55	100,00000	90,00000	80,00000	70,00000	60,00000	50,00000	40,00000	30,00252	21,10980	14,19563	9,11706	5,60216	3,30326	1,87626
a	0,6	100,00000	90,00000	80,00000	70,00000	60,00000	50,00000	40,00000	30,01404	21,26219	14,46096	9,44008	5,92563	3,58663	2,10051
t	0,65	100,00000	90,00000	80,00000	70,00000	60,00000	50,00000	40,00000	30,03380	21,41166	14,71236	9,74400	6,23132	3,85824	2,32038
u	0,7	100,00000	90,00000	80,00000	70,00000	60,00000	50,00000	40,00000	30,05967	21,55743	14,95144	10,03107	6,52212	4,11939	2,53533
r	0,75	100,00000	90,00000	80,00000	70,00000	60,00000	50,00000	40,00000	30,09046	21,69955	15,17871	10,30311	6,79823	4,37049	2,74496
i	0,8	100,00000	90,00000	80,00000	70,00000	60,00000	50,00000	40,00000	30,12525	21,83806	15,39616	10,56167	7,06211	4,61209	2,94982
t	0,85	100,00000	90,00000	80,00000	70,00000	60,00000	50,00000	40,00000	30,16341	21,97301	15,60362	10,80805	7,31371	4,84520	3,14959
y	0,9	100,00000	90,00000	80,00000	70,00000	60,00000	50,00000	40,00000	30,20373	22,10444	15,80301	11,04336	7,55549	5,07012	3,34429
	0,95	100,00000	90,00000	80,00000	70,00000	60,00000	50,00000	40,00000	30,24636	22,23236	15,99396	11,26855	7,78698	5,28714	3,53410
	1	100,00000	90,00000	80,00000	70,00000	60,00000	50,00000	40,00000	30,29034	22,35679	16,17769	11,48444	8,00939	5,49662	3,71918

Table 2. Evaluating the American options from the sample using the Bjerksund-Stendslandov (1993) model

		Asset price													
		50	60	70	80	90	100	110	120	130	140	150	160	170	180
	0,05	100,00000	90,00000	80,00000	70,00000	60,00000	50,00000	40,00000	30,00000	20,00000	10,17537	3,13141	0,47725	0,03334	0,00110
	0,1	100,00000	90,00000	80,00000	70,00000	60,00000	50,00000	40,00000	30,00000	20,00000	10,68140	4,30999	1,24605	0,25417	0,03717
T	0,15	100,00000	90,00000	80,00000	70,00000	60,00000	50,00000	40,00000	30,00000	20,00110	11,18646	5,17112	1,93365	0,58305	0,14337
i	0,2	100,00000	90,00000	80,00000	70,00000	60,00000	50,00000	40,00000	30,00000	20,04965	11,65041	5,86954	2,53929	0,94409	0,30431
m	0,25	100,00000	90,00000	80,00000	70,00000	60,00000	50,00000	40,00000	30,00000	20,14402	12,07379	6,46493	3,07945	1,30789	0,49888
e	0,3	100,00000	90,00000	80,00000	70,00000	60,00000	50,00000	40,00000	30,00000	20,26491	12,46203	6,98779	3,56752	1,66314	0,71235
	0,35	100,00000	90,00000	80,00000	70,00000	60,00000	50,00000	40,00000	30,00000	20,40066	12,82041	7,45614	4,01341	2,00560	0,93546
t	0,4	100,00000	90,00000	80,00000	70,00000	60,00000	50,00000	40,00000	30,00000	20,54434	13,15332	7,88169	4,42440	2,33399	1,16250
o	0,45	100,00000	90,00000	80,00000	70,00000	60,00000	50,00000	40,00000	30,00000	20,69174	13,46431	8,27250	4,80604	2,64827	1,38990
	0,5	100,00000	90,00000	80,00000	70,00000	60,00000	50,00000	40,00000	30,00000	20,84025	13,75623	8,63444	5,16257	2,94899	1,61546
m	0,55	100,00000	90,00000	80,00000	70,00000	60,00000	50,00000	40,00000	30,00000	20,98821	14,03140	8,97191	5,49736	3,23691	1,83781
a	0,6	100,00000	90,00000	80,00000	70,00000	60,00000	50,00000	40,00000	30,00276	21,13457	14,29174	9,28830	5,81309	3,51284	2,05612
t	0,65	100,00000	90,00000	80,00000	70,00000	60,00000	50,00000	40,00000	30,01390	21,27867	14,53882	9,58632	6,11196	3,77760	2,26989
u	0,7	100,00000	90,00000	80,00000	70,00000	60,00000	50,00000	40,00000	30,03210	21,42009	14,77398	9,86813	6,39579	4,03198	2,47887
r	0,75	100,00000	90,00000	80,00000	70,00000	60,00000	50,00000	40,00000	30,05606	21,55859	14,99835	10,13552	6,66610	4,27668	2,68295
i	0,8	100,00000	90,00000	80,00000	70,00000	60,00000	50,00000	40,00000	30,08470	21,69403	15,21289	10,38996	6,92418	4,51238	2,88210
t	0,85	100,00000	90,00000	80,00000	70,00000	60,00000	50,00000	40,00000	30,11716	21,82635	15,41844	10,63273	7,17113	4,73967	3,07639
y	0,9	100,00000	90,00000	80,00000	70,00000	60,00000	50,00000	40,00000	30,15273	21,95554	15,61573	10,86486	7,40789	4,95910	3,26589
	0,95	100,00000	90,00000	80,00000	70,00000	60,00000	50,00000	40,00000	30,19080	22,08162	15,80538	11,08730	7,63529	5,17117	3,45073
	1	100,00000	90,00000	80,00000	70,00000	60,00000	50,00000	40,00000	30,23091	22,20465	15,98797	11,30081	7,85406	5,37634	3,63104

Table 3. Evaluating the American options from the sample using the Bjerksund-Stenslandov (2002) model

		Asset price													
		50	60	70	80	90	100	110	120	130	140	150	160	170	180
	0,05	100,00000	90,00000	80,00000	70,00000	60,00000	50,00000	40,00000	30,00000	20,00000	10,18504	3,13842	0,47776	0,03335	0,00110
	0,1	100,00000	90,00000	80,00000	70,00000	60,00000	50,00000	40,00000	30,00000	20,00000	10,70275	4,32352	1,24882	0,25446	0,03719
T	0,15	100,00000	90,00000	80,00000	70,00000	60,00000	50,00000	40,00000	30,00000	20,00192	11,21543	5,19049	1,93969	0,58423	0,14353
i	0,2	100,00000	90,00000	80,00000	70,00000	60,00000	50,00000	40,00000	30,00000	20,05988	11,68517	5,89411	2,54901	0,94675	0,30488
m	0,25	100,00000	90,00000	80,00000	70,00000	60,00000	50,00000	40,00000	30,00000	20,16173	12,11320	6,49413	3,09293	1,31248	0,50015
e	0,3	100,00000	90,00000	80,00000	70,00000	60,00000	50,00000	40,00000	30,00000	20,28857	12,50527	7,02110	3,58470	1,66997	0,71462
	0,35	100,00000	90,00000	80,00000	70,00000	60,00000	50,00000	40,00000	30,00000	20,42911	12,86686	7,49311	4,03413	2,01486	0,93899
t	0,4	100,00000	90,00000	80,00000	70,00000	60,00000	50,00000	40,00000	30,00000	20,57670	13,20248	7,92190	4,44849	2,34576	1,16748
o	0,45	100,00000	90,00000	80,00000	70,00000	60,00000	50,00000	40,00000	30,00000	20,72732	13,51576	8,31560	4,83327	2,66257	1,39649
	0,5	100,00000	90,00000	80,00000	70,00000	60,00000	50,00000	40,00000	30,00000	20,87850	13,80963	8,68009	5,19272	2,96580	1,62377
m	0,55	100,00000	90,00000	80,00000	70,00000	60,00000	50,00000	40,00000	30,00000	21,02868	14,08646	9,01982	5,53021	3,25616	1,84790
a	0,6	100,00000	90,00000	80,00000	70,00000	60,00000	50,00000	40,00000	30,00535	21,17689	14,34819	9,33821	5,84842	3,53445	2,06803
t	0,65	100,00000	90,00000	80,00000	70,00000	60,00000	50,00000	40,00000	30,01939	21,32254	14,59644	9,63798	6,14958	3,80147	2,28363
u	0,7	100,00000	90,00000	80,00000	70,00000	60,00000	50,00000	40,00000	30,04016	21,46525	14,83258	9,92134	6,43550	4,05800	2,49442
r	0,75	100,00000	90,00000	80,00000	70,00000	60,00000	50,00000	40,00000	30,06638	21,60482	15,05775	10,19008	6,70771	4,30474	2,70028
i	0,8	100,00000	90,00000	80,00000	70,00000	60,00000	50,00000	40,00000	30,09703	21,74114	15,27295	10,44570	6,96752	4,54235	2,90117
t	0,85	100,00000	90,00000	80,00000	70,00000	60,00000	50,00000	40,00000	30,13127	21,87418	15,47901	10,68948	7,21604	4,77144	3,09714
y	0,9	100,00000	90,00000	80,00000	70,00000	60,00000	50,00000	40,00000	30,16841	22,00394	15,67670	10,92249	7,45421	4,99255	3,28826
	0,95	100,00000	90,00000	80,00000	70,00000	60,00000	50,00000	40,00000	30,20787	22,13047	15,86664	11,14566	7,68289	5,20618	3,47466
	1	100,00000	90,00000	80,00000	70,00000	60,00000	50,00000	40,00000	30,24920	22,25384	16,04942	11,35980	7,90280	5,41280	3,65646

Table 4. Evaluating the American options from the sample using the binominal model (n=350)

		Asset price													
		50	60	70	80	90	100	110	120	130	140	150	160	170	180
	0,05	100,00000	90,00000	80,00000	70,00000	60,00000	50,00000	40,00000	30,00000	20,00000	10,19905	3,14915	0,47988	0,03310	0,00108
	0,1	100,00000	90,00000	80,00000	70,00000	60,00000	50,00000	40,00000	30,00000	20,00000	10,73589	4,34628	1,25886	0,25574	0,03726
T	0,15	100,00000	90,00000	80,00000	70,00000	60,00000	50,00000	40,00000	30,00000	20,00773	11,26556	5,22452	1,95659	0,58774	0,14459
i	0,2	100,00000	90,00000	80,00000	70,00000	60,00000	50,00000	40,00000	30,00000	20,07851	11,74720	5,93846	2,57603	0,95751	0,30741
m	0,25	100,00000	90,00000	80,00000	70,00000	60,00000	50,00000	40,00000	30,00000	20,19513	12,18542	6,54787	3,12583	1,32843	0,50610
e	0,3	100,00000	90,00000	80,00000	70,00000	60,00000	50,00000	40,00000	30,00000	20,33354	12,58790	7,08335	3,62837	1,69249	0,72422
	0,35	100,00000	90,00000	80,00000	70,00000	60,00000	50,00000	40,00000	30,00000	20,48266	12,95861	7,56305	4,07923	2,03889	0,95270
t	0,4	100,00000	90,00000	80,00000	70,00000	60,00000	50,00000	40,00000	30,00000	20,64152	13,29634	7,99880	4,50425	2,38003	1,18397
o	0,45	100,00000	90,00000	80,00000	70,00000	60,00000	50,00000	40,00000	30,00000	20,79536	13,61625	8,39877	4,89911	2,69582	1,41589
	0,5	100,00000	90,00000	80,00000	70,00000	60,00000	50,00000	40,00000	30,00000	20,95667	13,91870	8,76892	5,26463	3,01202	1,65048
m	0,55	100,00000	90,00000	80,00000	70,00000	60,00000	50,00000	40,00000	30,00000	21,11104	14,20110	9,11371	5,60535	3,30652	1,87639
a	0,6	100,00000	90,00000	80,00000	70,00000	60,00000	50,00000	40,00000	30,01409	21,26105	14,46596	9,43666	5,92498	3,58333	2,10171
t	0,65	100,00000	90,00000	80,00000	70,00000	60,00000	50,00000	40,00000	30,03241	21,41378	14,71539	9,74051	6,22770	3,86069	2,32419
u	0,7	100,00000	90,00000	80,00000	70,00000	60,00000	50,00000	40,00000	30,05792	21,56030	14,95113	10,02750	6,52274	4,12527	2,53458
r	0,75	100,00000	90,00000	80,00000	70,00000	60,00000	50,00000	40,00000	30,09084	21,70058	15,17562	10,29946	6,80229	4,37606	2,74487
i	0,8	100,00000	90,00000	80,00000	70,00000	60,00000	50,00000	40,00000	30,12435	21,83530	15,39544	10,55796	7,06798	4,61475	2,95430
t	0,85	100,00000	90,00000	80,00000	70,00000	60,00000	50,00000	40,00000	30,16074	21,97351	15,60558	10,80426	7,32124	4,84284	3,15380
y	0,9	100,00000	90,00000	80,00000	70,00000	60,00000	50,00000	40,00000	30,20302	22,10717	15,80654	11,03948	7,56311	5,06833	3,34467
	0,95	100,00000	90,00000	80,00000	70,00000	60,00000	50,00000	40,00000	30,24708	22,23600	15,99896	11,26460	7,79462	5,29013	3,52932
	1	100,00000	90,00000	80,00000	70,00000	60,00000	50,00000	40,00000	30,29083	22,36017	16,18358	11,48044	8,01666	5,50280	3,72087

Table 5. Evaluating the American options from the sample using the trinominal model (n=100)

		Asset price													
		50	60	70	80	90	100	110	120	130	140	150	160	170	180
	0,05	100,00000	90,00000	80,00000	70,00000	60,00000	50,00000	40,00000	30,00000	20,00000	10,19825	3,14746	0,47911	0,03304	0,00103
	0,1	100,00000	90,00000	80,00000	70,00000	60,00000	50,00000	40,00000	30,00000	20,00000	10,73506	4,34383	1,25827	0,25513	0,03699
T	0,15	100,00000	90,00000	80,00000	70,00000	60,00000	50,00000	40,00000	30,00000	20,00346	11,25985	5,22140	1,95855	0,58726	0,14413
i	0,2	100,00000	90,00000	80,00000	70,00000	60,00000	50,00000	40,00000	30,00000	20,07530	11,74857	5,93480	2,56940	0,95335	0,30810
m	0,25	100,00000	90,00000	80,00000	70,00000	60,00000	50,00000	40,00000	30,00000	20,19105	12,18049	6,54370	3,13005	1,32292	0,50548
e	0,3	100,00000	90,00000	80,00000	70,00000	60,00000	50,00000	40,00000	30,00000	20,33235	12,58898	7,07865	3,62876	1,69367	0,72438
	0,35	100,00000	90,00000	80,00000	70,00000	60,00000	50,00000	40,00000	30,00000	20,47753	12,95696	7,55781	4,07649	2,03364	0,95214
t	0,4	100,00000	90,00000	80,00000	70,00000	60,00000	50,00000	40,00000	30,00000	20,63658	13,29196	7,99310	4,50051	2,38195	1,18100
o	0,45	100,00000	90,00000	80,00000	70,00000	60,00000	50,00000	40,00000	30,00000	20,78916	13,60944	8,39265	4,89876	2,70119	1,41891
	0,5	100,00000	90,00000	80,00000	70,00000	60,00000	50,00000	40,00000	30,00000	20,95150	13,91357	8,76237	5,26711	2,99895	1,64913
m	0,55	100,00000	90,00000	80,00000	70,00000	60,00000	50,00000	40,00000	30,00000	21,10956	14,19801	9,10670	5,61026	3,30600	1,87051
a	0,6	100,00000	90,00000	80,00000	70,00000	60,00000	50,00000	40,00000	30,00000	21,25993	14,46479	9,42915	5,93174	3,59367	2,10467
t	0,65	100,00000	90,00000	80,00000	70,00000	60,00000	50,00000	40,00000	30,01188	21,40273	14,71600	9,73249	6,23434	3,86434	2,32448
u	0,7	100,00000	90,00000	80,00000	70,00000	60,00000	50,00000	40,00000	30,04478	21,55168	14,95357	10,01905	6,52075	4,12035	2,53260
r	0,75	100,00000	90,00000	80,00000	70,00000	60,00000	50,00000	40,00000	30,08104	21,69705	15,17882	10,29062	6,79247	4,36384	2,73814
i	0,8	100,00000	90,00000	80,00000	70,00000	60,00000	50,00000	40,00000	30,11786	21,83681	15,39303	10,54880	7,05161	4,60438	2,95212
t	0,85	100,00000	90,00000	80,00000	70,00000	60,00000	50,00000	40,00000	30,15865	21,97092	15,59722	10,79485	7,30321	4,84506	3,15566
y	0,9	100,00000	90,00000	80,00000	70,00000	60,00000	50,00000	40,00000	30,20119	22,09973	15,79234	11,02985	7,54903	5,07466	3,34992
	0,95	100,00000	90,00000	80,00000	70,00000	60,00000	50,00000	40,00000	30,24326	22,22363	15,97915	11,25474	7,78412	5,29440	3,53604
	1	100,00000	90,00000	80,00000	70,00000	60,00000	50,00000	40,00000	30,28451	22,34286	16,16452	11,47026	8,00937	5,50504	3,71516

```
» [p,table,stats] = friedman(Y)

p =

     0


table =

    'Source'     'SS'                         'df'     'MS'                              'Chi-sq'                     'Prob>Chi-sq'
    'Columns'    [1.026192307692308e+003]     [  4]    [2.565480769230769e+002]          [4.152684824902723e+002]     [           0]
    'Error'      [2.588076923076924e+002]     [516]    [        0.50156529516995]                              []               []
    'Total'      [                   1285]    [649]                                []                          []               []


stats =

       source: 'friedman'
            n: 130
    meanranks: [4.84230769230769 3.85000000000000 1.32692307692308 2.01153846153846                 ]
        sigma: 1.57199040905275
```

Fig. 2. Results of the Friedman test for the error sample obtained by the Bjerksund-Stensland (1993) model, Bjerksund-Stensland (2002) model, binominal and trinominal model in evaluating American options

5 CONCLUSION

Taking into account the development of computer technology, i.e. architecture improvements and the increased speed of the new computer models, it is clear that the calculation accuracy of numerical methods in the same time period will be significantly higher on the modern computers than it was at the time when Bjerksund-Stensland models were published. The results of this study confirmed our assumptions and proved that the numerical methods provide a greater precision of calculations when compared to the Bjerksund-Stensland model if the computation time is limited to one second. Out of the set of numerical methods presented for the evaluation of plain vanilla American options, it was the binomial model that proved to be the most precise, followed by the trinomial model.

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