The Matrix Pencil Technique for Three-Dimensional Frequency Estimation

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***Abstract*—This paper deals with the problem of estimating the three-dimensional (3-D) frequency using Matrix Pencil (MP) technique. A signal modeled by the sum of 3-D complex exponentials is considered and then a MP method is applied directly to this signal on a snapshot-by-snapshot basis and hence is computationally quite efficient to estimate the 3-D frequency with high resolution. Non-stationary in the data then has a little effect for this method, as no assumption is made about the statistics of the environment. It is applied directly to the single 3-D data snapshot without forming a covariance matrix and operates in two mainly steps; first, three proposed matrices are constructed from the single 3-D data snapshot, and then apply the MP approach for each matrix to extract out the 3-D frequency efficiently. Furthermore, the proposed technique is still operational when there exist identical frequencies in one or more dimensions. Limited numerical examples are presented to illustrate the performance and accuracy of the proposed technique.**

Keywords-Three-dimensional; Matrix Pencil; Eigenvectors; Eigenvalues.

I. INTRODUCTION

Multidimensional frequency estimation plays an important role in our proposed interference mitigation framework. In a variety of multidimensional statistical signal processing applications such as spectral estimation, texture image modeling, and classification, it is often desired to estimate multidimensional frequencies. The existing 1 and 2-D frequency estimation methods can be divided into two classes: namely scanning methods and analytical ones. The scanning methods consist in scanning the frequency space with a discrete finite lag frequency and estimate the harmonics frequency using the peak picking technique in an appropriate pseudo spectrum [1]. The analytical methods, called also high resolution (HR) techniques or subspace approaches, are based on the decomposition of the space spanned by the eigenvectors of the data (or autocorrelation) matrix into two orthogonal subspaces namely noise and signal subspaces. This class contains the Pisarenko method, root-MUSIC, SEPRIT [2-4], and MP method [5-7]. A modified MP technique for 2-D complex exponential estimation is presented in [8]. In this technique, easy different steps from that given in [9] to estimate the 3-D frequency are proposed. From the computational point of view, the HR methods are computationally efficient since they do not require searching over the entire 2-D frequency space to locate the harmonic peaks as the scanning method does. The HR methods have been extended to 3-D spectral estimation [10-14].

In this paper, a MP method based on the technique described in [11] to accurately estimate the 3-D frequency efficiently is proposed. This technique is applied directly to the data on a snapshot-by-snapshot basis and hence is computationally quite efficient. Non-stationary in the data then has a little effect for this method, as no assumption is made about the statistics of the environment.

This paper is organized as follows: Section II presents the sum of the 3-D complex exponential model. In section III, the procedures of the proposed technique to estimate the 3-D frequency are presented. In section VI, numerical examples are presented to compute the performance and accuracy of the proposed technique and finally the conclusions are introduced in section V.

II. PROBLEM FORMULATION

A set of 3-D data $\{ y(m,n,t) \}$, $1 \le m \le M$, $1 \le n \le N$, $1 \le t \le T$, represent by a $M \times N \times T$ cube, where $y(m,n,t)$ is a scalar one modeled as a sum of K, 3-D complex exponential, signals corrupted by noise as

$$y(m,n,t) = x(m,n,t) + w(m,n,t) , \qquad (1)$$

where the noiseless data are modeled as

$$x(m,n,t) = \sum_{k=1}^{K} a_k \exp\left[j2\pi\left(f_{1k}m + f_{2k}n + f_{3k}t \right) + j\varphi_k \right] \qquad (2)$$

The triplets $\left(f_{1k}, f_{2k}, f_{3k} \right)$ are the 3-D normalized frequencies, the parameters a_k and φ_k are respectively the amplitude and phase of the k^{th} signal. The process $x(m,n,t)$ is stationary. The additive noise $w(m,n,t)$ is assumed to be a 3-D white Gaussian process uncorrelated with the $x(m,n,t)$. The basic problem here is to estimate the normalized frequencies $\left(f_{1k}, f_{2k}, f_{3k} \right)$ from the noisy observed cubic data $y(m,n,t)$.

III. PROPOSED METHOD PROCEDURES

The following is a step by step description of what needs to be done to obtain the normalized frequencies (f_{1k}, f_{2k}, f_{3k}), $1 \le k \le K$, from the noisy observed cubic data $y(m,n,t)$:

A. Three matrices construction

First three proposed different matrices to solve the problem stated in section II are arranged as follows

$$[Y_1] = \begin{bmatrix} y(1,1,1) & y(2,1,1) & \cdots & y(M,1,1) \\ y(1,1,2) & y(2,1,2) & \cdots & y(M,1,2) \\ \vdots & \vdots & \iddots & \vdots \\ y(1,1,T) & y(2,1,T) & \cdots & y(M,1,T) \\ y(1,2,1) & y(2,2,1) & \cdots & y(M,2,1) \\ y(1,2,2) & y(2,2,2) & \cdots & y(M,2,2) \\ \vdots & \vdots & \iddots & \vdots \\ y(1,2,T) & y(2,2,T) & \cdots & y(M,2,T) \\ \vdots & \vdots & \iddots & \vdots \\ y(1,N,T) & x(2,N,T) & \cdots & y(M,N,T) \end{bmatrix}^T \tag{3}$$

$$[Y_2] = \begin{bmatrix} y(1,1,1) & y(1,2,1) & \cdots & y(1,N,1) \\ y(1,1,2) & y(1,2,2) & \cdots & y(1,N,2) \\ \vdots & \vdots & \iddots & \vdots \\ y(1,1,T) & y(1,2,T) & \cdots & y(1,N,T) \\ y(2,1,1) & y(2,2,1) & \cdots & y(2,N,1) \\ y(2,1,2) & y(2,2,2) & \cdots & y(2,N,2) \\ \vdots & \vdots & \iddots & \vdots \\ y(2,1,T) & y(2,2,T) & \cdots & y(2,N,T) \\ \vdots & \vdots & \iddots & \vdots \\ y(M,1,T) & y(M,2,T) & \cdots & y(M,N,T) \end{bmatrix}^T \tag{4}$$

$$[Y_3] = \begin{bmatrix} y(1,1,1) & y(1,1,2) & \cdots & y(1,1,T) \\ y(1,2,1) & y(1,2,2) & \cdots & y(1,2,T) \\ \vdots & \vdots & \iddots & \vdots \\ y(1,N,1) & y(1,N,2) & \cdots & y(1,N,T) \\ y(2,1,1) & y(2,1,2) & \cdots & y(2,1,T) \\ y(2,2,1) & y(2,2,2) & \cdots & y(2,2,T) \\ \vdots & \vdots & \iddots & \vdots \\ y(2,N,1) & y(2,N,2) & \cdots & y(2,N,T) \\ \vdots & \vdots & \iddots & \vdots \\ y(M,N,1) & y(M,N,2) & \cdots & y(M,N,T) \end{bmatrix}^T \tag{5}$$

B. Estimating the number of signals

Second, the number of signals K is estimated by obtaining the eigenvalues of the matrix ($[Y][Y]^H$) where matrix $[Y]$ is any one of the three matrices and the superscript H denotes the conjugate transpose of a matrix. If we consider that the data $y(m,n,t)$ are not contaminated by any noise only the first K eigenvalues are non zero. In the presence of noise, the parameter K is estimated by observing the ratio of the various singular values to the largest one. Consider the singular value λ_c such that

$$\frac{\lambda_c}{\lambda_{\max}} \approx 10^{-b} \tag{6}$$

where b is the number of significant decimal digits in the data $y(m,n,t)$. The singular values for which the ratio in (6) is bellow 10^{-b} are essentially noise singular values, and they should not be used.

C. Estimating 3-D frequency (f_{1k}, f_{2k}, f_{3k})

To estimate the first component of the 3-D frequencies (f_{1k});

i- The matrix $[U]$ whose columns are the eigenvectors of ($[Y_1][Y_1]^H$) is obtained.

ii- Next, construct the following sub matrix based on the K dominant eigenvalues:
$[U_s] \equiv$ the K (corresponding to the dominate eigenvalues) columns of $[U]$.

iii- Next, construct the following matrices:
$[U_1] = [U_s]$ with the last row deleted.
$[U_2] = [U_s]$ with the first row deleted.

iv- Finally, form the matrix pencil

$$[U_2] - \sigma[U_1] = 0 \tag{7}$$

and the f_{1k} components can be estimated by obtaining the angle of the eigenvalues of the matrix $[U_1]^\Psi [U_2]$, i.e.

$$f_{1k} = \frac{1}{2\pi}\mathrm{Im}[\ln(\sigma_k)]; \quad 1 \le k \le K \tag{8}$$

The superscript Ψ is the Moore-Penrose pseudo-inverse of a matrix and is defined by

$$[U_1]^\Psi = \left\{[U_1]^H [U_1]\right\}^{-1} [U_1]^H \tag{9}$$

To estimate the other 3-D frequencies f_{2k} and f_{3k}, The steps from i to iv are repeated using the matrices $[Y_2]$ and $[Y_3]$, respectively.

IV. NUMERICAL SIMULATIONS

In this section we present numerical examples to demonstrate the validity of the proposed method. For simplicity a 2×2×2 noiseless data set according to the model in (2) is generated. The signal parameters are given in Table I.

The number of signals K is estimated by obtaining the eigenvalues of ($[Y][Y]^H$)

$$\text{Eigenvalues of } ([Y][Y]^H) = \begin{bmatrix} -0.910 - j0.745 \\ 0 \end{bmatrix}$$

TABLE I. THE SIGNAL PARAMETERS.

Signal number	**Signal amplitude**	**Signal phase**	f_{1k}	f_{2k}	f_{3k}
$k = 1$	1 V/m	0.5 radian	0.10 Hz	0.15 Hz	0.20 Hz

It is noticed that the first eigenvalue is non-zero which means that only one signal is considered and the actual and estimated 3-D frequencies of this signal after applying the proposed technique are equal, because the noiseless data case is considered here, which means that the proposed method yields the exact 3-D frequencies estimation.

Now the case of data embedded with Gaussian noise is considered. The accepted estimated 3-D frequency to investigate the effect of the noise at Signal-to-Noise Ratio (SNR) equal to 15dB and 20dB using the proposed technique are given in Table II and III, respectively. For each value of the SNR, ten trials are used.

TABLE II. 3-D FREQUENCY ESTIMATED.

SNR	$\hat{f}_{11}$	$\hat{f}_{21}$	$\hat{f}_{31}$
15dB	0.071 Hz	0.131 Hz	0.189 Hz

TABLE III. 3-D FREQUENCY ESTIMATED.

SNR	$\hat{f}_{11}$	$\hat{f}_{21}$	$\hat{f}_{31}$
20dB	0.096 Hz	0.146 Hz	0.198 Hz

To estimate the 3-D frequency for two signals a 3×3×3 data set embedded with Gaussian noise with SNR = 15dB is considered with the parameters given in Table IV.

TABLE IV. THE SIGNAL PARAMETERS.

Signal number	**Signal amplitude**	**Signal phase**	f_{1k}	f_{2k}	f_{3k}
$k = 1$	1 V/m	0.5 radian	0.10 Hz	0.15 Hz	0.20 Hz
$k = 2$	1.5 V/m	0.25 radian	0.30 Hz	0.35 Hz	0.40 Hz

Again the number of signals K is estimated by obtaining the eigenvalues of $([Y][Y]^H)$

$$\text{Eigenvalues of } ([Y][Y]^H) = \begin{bmatrix} 59.025 \\ 19.855 \\ 0.197 \end{bmatrix}$$

It is obvious that the first and second eigenvalues are the signal eigenvalues while the third one is the noise eigenvalue and is excluded which means that there are two signals. The accepted estimated 3-D frequency using the proposed technique is given in Table V. The average of ten trials is used.

TABLE V. 3-D FREQUENCY ESTIMATED.

k	**SNR**	$\hat{f}_{1k}$	$\hat{f}_{2k}$	$\hat{f}_{3k}$
1	15dB	0.105 Hz	0.153 Hz	0.203 Hz
2	15dB	0.301 Hz	0.351 Hz	0.399 Hz

The previous examples illustrate that as the dimension size is increased the error between the actual and the estimated frequencies is decreased and to estimate the 3-D frequency of K signals, the size of each dimension must equal to $K + 1$, i.e.

$$M = N = T = K + 1 \tag{10}$$

Finally, to demonstrate the performance measured by Root Mean Square Error (RMSE) versus SNR of the proposed technique for 3-D identical frequency estimation of two signals a 3×3×3 data set embedded with Gaussian noise with SNR varied from 0 dB to 55 dB is considered with the parameters given in Table VI.

TABLE VI. THE SIGNAL PARAMETERS.

Signal number	**Signal amplitude**	**Signal phase**	f_{1k}	f_{2k}	f_{3k}
$k = 1$	1 V/m	0.0 radian	0.10 Hz	0.15 Hz	0.10 Hz
$k = 2$	1 V/m	0.0 radian	0.30 Hz	0.35 Hz	0.30 Hz

Notice that there are identical frequencies in two dimensions, which is a case that some other algorithms fail to deal with. Fig. 1 depicts the RMSE versus SNR. The RMSE results are averaged over all frequencies and obtained through 1000 realizations.

V. CONCLUSIONS

A MP method to estimate 3-D frequencies of signals embedded in white Gaussian noise is proposed in this paper. This method is applied directly to the data on a snapshot-by-snapshot basis and hence its computational is quite efficient. Non-stationary in the data then has little effect for this method, as no assumption is made about the statistics of the environment. To estimate 3-D frequencies of more signals we must increase the number of the 3-D data. It is shown that this technique remains operational when there exist identical frequencies in one or more dimensions. Accepted and accurate results are observed through simulated numerical examples.

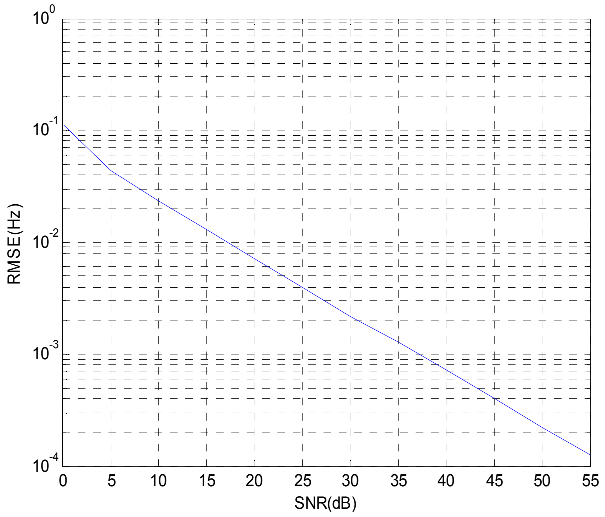

Fig. 1 RMSE of the proposed technique versus SNR.

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