A TECHNICAL NOTE ON SOLVING LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

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ABSTRACT

In this paper, the form of a general solution to certain general nth-order linear differential equations of constant coefficients of the form

$$\sum_{j=0}^{n}\left[\binom{n-1}{j}a^{j}+\binom{n-1}{j}a^{j-1}b\right]y^{(n-j)} = f(x) \qquad (1)$$

is established .

Key words: General Solution, nth-Order Linear Differential Equations.

1. INTRODUCTION

The problem of finding a general solution of the nth-order linear differential equation of constant coefficients $L[y]=f$, was considered by Ardaletova and Kydraliev [1] by defending an equivalent differential operator $M(y)=f$. Richard and his colleagues [2] have considered a specific type of the same equation by using different technique, and have given the form of a general solution. Here we are using the same technique given in [2] to find a general solution form to equation (1).

2. MAIN RESULTS

In this section we have introduced our main results.

Theorem (1)

A general solution to equation (1) where n is a positive integer, a and b are constants, is given by.

$$y = e^{-ax}\int\left(\ldots\left(\ldots\int e^{(a-b)x}\left(\int e^{bx} f(x)dx\right)dx\right)\ldots\right)dx \qquad (2)$$

Proof:

Since equation (1) can be written as

$$\left(\sum_{j=0}^{n-1}\binom{n-1}{j}a^{j}y^{(n-j-1)}\right)' + b\left(\sum_{j=0}^{n-1}\binom{n-1}{j}a^{j}y^{(n-j-1)}\right) = f(x) \qquad (3)$$

So, it has

$$\sum_{j=0}^{n-1}\binom{n-1}{j}a^{j}y^{(n-j-1)} = e^{-bx}\int e^{bx} f(x)dx \qquad (4)$$

as a general solution. Now, we shall use induction on n, to show that (2) is true.

For n=2, equation (4) is reduced to

$$y' + ay = e^{-bx}\int e^{bx} f(x)dx \qquad (5)$$

Hence, equation (5) has

$$y = e^{-ax}\int e^{(a-b)x}\left(\int e^{bx} f(x)dx\right)dx \tag{6}$$

as a general solution . Let the result be true for the integer n, and that

$$\sum_{j=0}^{n}\binom{n}{j}a^{j}y^{(n-j)} = e^{-bx}\int e^{bx} f(x)dx \tag{7}$$

Now (7) can be written as

$$\left(\sum_{j=0}^{n-1}\binom{n-1}{j}a^{j}y^{(n-j-1)}\right)' + a\left(\sum_{j=0}^{n-1}\binom{n-1}{j}a^{j}y^{(n-j-1)}\right) = e^{-bx}\int e^{bx} f(x)dx \tag{8}$$

and from the case $n = 2$, a general solution of (8) may be given by

$$\sum_{j=0}^{n-1}\binom{n-1}{j}a^{j}y^{(n-j-1)} = e^{-ax}\int e^{(a-b)x}\left(\int e^{bx} f(x)dx\right)dx. \tag{9}$$

Using induction hypothesis, we have

$$y = e^{-ax}\int\left(\int\left(\ldots\left(\int e^{(a-b)x}\left(\int e^{bx} f(x)dx\right)dx\right)\ldots\right)dx\right)dx$$

as a general solution of (7) . This completes the proof.

The following corollary deals with the homogeneous case and it arises naturally from the theorem (1).

Corollary (1) Let a and b be constants, a≠b, then a general solution of the following homogeneous equation

$$\sum_{j=0}^{n}\left[\binom{n-1}{j}a^{j} + \binom{n-1}{j-1}a^{j}b\right]y^{(n-j)} = 0, \tag{10}$$

is given by,

$$y = \frac{c_1 e^{-bx}}{(a-b)^{n-1}} + e^{-ax}\sum c_j x^{n-j}. \tag{11}$$

<u>Proof</u>

To prove the result, we must show that (10) satisfies (9). For this, multiply (10) by e^{ax} we have

$$e^{ax}y = \frac{c_1 e^{(a-b)x}}{(a-b)^{n-1}} + \sum_{j=2}^{n} c_j x^{n-j}. \tag{12}$$

Taking $(n-1)$ derivatives of (12) one can easily have

$$\left(e^{ax}y\right)^{n-1} = c_1 e^{(a-b)x}, \tag{13}$$

using (13), we have,

$$c_1 e^{(a-b)x} = \sum_{j=0}^{n-1}\binom{n-1}{j}\left(e^{ax}\right)^{(j)} y^{(n-1-j)} = e^{ax}\sum_{j=0}^{n-1}\binom{n-1}{j}a^{j}y^{(n-1-j)}, \tag{14}$$

or

$$c_1 e^{-bx} = \sum_{j=0}^{n-1} \binom{n-1}{j} a^j y^{(n-1-j)}. \tag{15}$$

Differentiate in (15), we have

$$-bc_1 e^{-bx} = \sum_{j=0}^{n-1} \binom{n-1}{j} a^j y^{(n-j)} = \sum_{j=0}^{n} \binom{n-1}{j} a^j y^{(n-j)} = A, say, \tag{16}$$

Using the fact that

$$\sum_{j=0}^{n-1} \binom{n-1}{j} a^j y^{(n-1-j)} = \sum_{j=0}^{n} \binom{n-1}{j-1} a^{j-1} y^{(n-j)} = \sum_{j=0}^{n} \binom{n-1}{j-1} a^{j-1} y^{(n-j)}. \tag{17}$$

Hence, we have,

$$c_1 b e^{-bx} = \sum_{j=0}^{n} \binom{n-1}{j-1} a^{j-1} b\, y^{(n-j)} = B, say \tag{18}$$

Equation (15) has been used. Now, from (16) and (18) we have,

$$\sum_{j=0}^{n} \left[\binom{n-1}{j} a^j + \binom{n-1}{j-1} a^{j-1} b \right] y^{(n-j)} = 0.$$

This completes the proof.

REMARK: We remark that the relation between the coefficients of the second and the third term of (1) may be used to identify the problem under consideration is either of type (1) or not. These relations are given by

$$\binom{n}{n-2} a^2 - (n-1) m_1 a + m_2 = 0 \tag{19}$$

And

b= $m_1 - (n-1)a$

We can end this article by the following example

Example:

Find a general solution of

$$y''' + \frac{4}{3} y'' + \frac{7}{12} y' + \frac{1}{12} y = 0 \tag{20}$$

Solution

Here, $m_1 = \frac{4}{3}, m_2 \frac{7}{12}, and\ n = 3.$

Substituting these quantities into equation (19), we get

$$3a^2 - \frac{8}{3} a + \frac{7}{12} = 0. \tag{21}$$

This implies that

$a = \frac{1}{2} or \frac{7}{18}$, and consequently $b = \frac{1}{3} or \frac{5}{9}$, respectively. Now for $a = \frac{1}{2} and \quad b = \frac{1}{3}$, equation (19) can be written as

$$\left(y'' + y' + \frac{1}{4}y\right)' + \frac{1}{3}\left(y'' + y' + \frac{1}{4}y\right) = 0,$$

and therefore, a general solution of (19) may be given by

$$y = c_1 e^{-\frac{1}{3}x} + e^{-\frac{1}{2}x}(c_2 x + c_3).$$

References

[1] A.B. Urdaletova and S.K Kydyraliev, Solving Linear Differential Equations by operator Factorization, The College Mathematics Journal, VOL.27, No.3, May 1996.

[2] Richard E. Bayne m and others, A. Note on Linear Differential Equations With Constant Coefficients, Missouri Journal of Mathematical Sciences Articles, VOL.10, NO.1, Winter 1998.