On some kinds of fuzzy connected space

by

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Abstract

In this paper new results in fuzzy connected space are obtaind and prove that locally fuzzy connectedness is good extension of locally connectedness also it is proved that in a T_1-fuzzy compact space the notations c-zero dimensional, Strong c-zero dimensional and Totally c_i-connected are equivalents.

Keywords :Fuzzy connected space, fuzzy strong connected, fuzzy super connected, c-zero dimensional, strong c-dimensional and totally c_i-connected.
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1 Introduction

In 1965 Zadeh in his classical paper generalized characteristic functions to fuzzy sets. Chang in 1968 introduced the topological structure fuzzy sets in a given set. Pu and Liu defined connectedness by using the concept fuzzy closed set. Lowen also defined an extension of a connectedness in a restricted family of fuzzy topologies. Fatteh and Bassam studied further the notion of fuzzy super connected and fuzzy strongly connected spaces. However they defined connectedness only for a crisp set of a fuzzy topological space. Ajmal and Kohli extend the notations of connectedness to an arbitrary fuzzy set, and defined the notions of c-zero dimensional, total disconnectedness and strongly zero connectedness.

In this paper we give more results on these spaces and prove that locally fuzzy connectedness is a good extension of locally connectedness. Also, it is proved that in a T_1-fuzzy compact space the notions c-zero dimensional, Strong c-zero dimensional and totally c_i-connected are equivalents.

2 Basic definitions

The following definitions have been used to obtain the results and properties developed in this paper.

Definition 1 [7]*A fuzzy topological space X is said to be fuzzy connected if it has no proper fuzzy clopen set. (A fuzzy set λ in X is proper if $\lambda \neq 0$ and $\lambda \neq 1$).*

Definition 2 [7]*A fuzzy topological space X is said to be fuzzy super- connected iff X does not have non- zero fuzzy open sets λ and μ such that $\lambda + \mu \leq 1$.*

Definition 3 [7]*A fuzzy topological space X is said to be fuzzy strongly connected if it has no non- zero fuzzy closed sets f and k such that $f + k \leq 1$.*

Definition 4 [1]*A fuzzy topological space X is said to be c-zero dimensional if every crisp fuzzy point x_1 in X and every fuzzy open set μ containing x_1 there exists a crisp clopen fuzzy set δ in X such that $x_1 \leq \delta \leq \mu$.*

Definition 5 [1]*A fuzzy topological space X is said to be totally c_i-disconnected $(i = 1, 2, 3, 4)$ if the support of every non zero c_i-connected fuzzy set in X is a singleton.*

Definition 6 [1]*A fuzzy topological space X is said to be strongly c-zero dimensional if it is not c-connected between any pair of its disjoint fuzzy closed sets.*

Definition 7 [2]*A fuzzy topological space X is said to be fuzzy locally connected at a fuzzy point x_α in X if for every fuzzy open set μ in X containing x_α, there exists a connected fuzzy open set δ in x such that $x_1 \leq \delta \leq \mu$.*

Definition 8 *A fuzzy topological space (X,T) is said to be locally fuzzy super connected (locally fuzzy strong connected) at a fuzzy point x_a in X if for every fuzzy open set μ in X containing x_a there exist a fuzzy super connected (fuzzy strong connected) open set η in X such that $x_a \leq \eta \leq \mu$*

Definition 9 *A fuzzy quasi-component of a fuzzy point x_1 in a fuzzy topological space (X,T) is the smallest fuzzy clopen subset of X contain x_1 and denoted by Q.*

Definition 10 *A fuzzy path component of a fuzzy point x_1 in a fuzzy topological space (X,T) is the maximal fuzzy path connected in (X,T) contain x_1 and denoted by C.*

3 Fuzzy connectedness and its stronger forms

In this section we study some stronger forms of connectedness such as fuzzy super connected, totally c_i-disconnected and fuzzy strongly connected introduced by Fatteh and Bassam and we proved that locally fuzzy connectedness is a good extension of locally connectedness also we get some additional results and properties for these spaces.

3.1 Theorem

A topological space (X, τ) is locally connected if and only if $(X, \omega(\tau))$ is locally connected.

Proof. Let μ be a fuzzy open set in $\omega(\tau)$ containing a fuzzy point x_α, since μ is lower semicontinuous function, then by local connectedness of (X, τ) there exists an open connected set U in X containing x and contained in the support of μ. i.e $(x \in U \subset Supp\ \mu)$. Now χ_U is the characteristic function of U and it is a lower semicontinuous, then $\chi_U \wedge \mu$ is fuzzy open set in $\omega(\tau)$.We claim $\delta = \chi_U \wedge \mu$ is fuzzy connected set containing x_α, if not then by [8, Th.(3.1),] there exists a non zero lower semicontinuous functions μ_1, μ_2 in $\omega(\tau)$ such that

$$\mu_1 \mid \delta + \mu_2 \mid \delta = 1$$

Now $Supp\ \delta = U$ and $Supp\ \mu_1$, $Supp\ \mu_2$ are open sets in τ such that

$$U \subset Supp\ \mu_1 \cup\ Supp\ \mu_2$$

then,

$$U \cap Supp\ \mu_1 \neq \Phi$$

and

$$U \cap\ Supp\ \mu_2 \neq \Phi$$

then

$$(U \cap Supp\ \mu_1)\ \cup (U \cap\ Supp\ \mu_2) = U \cap (Supp\ \mu_1 \cup Supp\ \mu_2) = U$$

is not connected. Conversely let U be an open set in τ containing x, $x_\alpha \in \chi_U$, (χ_u is the characteristic function of U), χ_u is fuzzy open set in $\omega(\tau)$. By fuzzy connectedness of $(X, \omega(\tau))$ there exists a fuzzy open connected set μ in $\omega(\tau)$ such that.

$$x_\alpha \leq \mu \leq \chi_U$$

Claim that $Supp\ \mu$ is connected $(x \in Supp\ \mu \subset U)$, if not there exists two non empty open sets $G_1, G_2 \in \tau$ such that

$$Supp\mu = G_1 \cup G_2 \text{ and } G_1 \cap G_2 = \Phi$$

it is clear that

$$\chi_{G_1} +\ \chi_{G_2} = 1_\mu$$

which is contradiction, because μ is fuzzy connected. ∎

3.2 Theorem

If G is a subset of a fuzzy topological space (X,T) such that μ_G (μ_G is the characteristic function of a subset G of X)is fuzzy open in X, then if X is super connected space implies G is fuzzy super connected space.

Proof. Suppose that G is not fuzzy super connected space then by [7, Th.(6.1),] exists fuzzy open sets λ_1, λ_2 in X such that $\lambda_1/G \neq 0, \lambda_2/G \neq 0$ and $\lambda_1/G + \lambda_2/G \leq 1$ there for $\lambda_1 \wedge \mu_G + \lambda_2 \wedge \mu_G \leq 1$.
Then X is not fuzzy super connected space and we get contradiction. ■

3.3 Theorem

If A and B are fuzzy strong connected subsets of a fuzzy topological space (X,T) and $\overline{\mu_B}/A \neq 0$ or $\overline{\mu_A}/B \neq 0$, then $A \vee B$ is a fuzzy strong connected subset of X where μ_A, μ_B are the characteristic function of a subset A and A and B respectively.

Proof. Suppose $Y = A \vee B$ is not strong fuzzy connected subset of X. Then there exist fuzzy closed sets δ and λ such that $\delta/Y \neq 0$ and $\lambda/Y \neq 0$ and $\delta/Y + \lambda/Y \leq 1$. Since A is fuzzy strong connected subset of X, then either $\delta/A = 0$ or $\lambda/A = 0$. Without loss the generality assume that $\delta/A = 0$ in this case since B is also fuzzy strong connected we have

$$\delta/A = 0, \lambda/A \neq 0, \delta/B \neq 0, \lambda/B = 0$$

there for

$$\lambda/A + \overline{\mu_B}/A \leq 1 \tag{1}$$

If $\overline{\mu_B}/A \neq 0$, then $\lambda/A \neq 0$ with (1) imply that A is not fuzzy connected subset of X. In the same way if $\overline{\mu_A}/B \neq 0$ then $\delta/B \neq 0$ and $\lambda/B + \overline{\mu_A}/B \leq 1$ imply that B is not a fuzzy strong connected subset of X, we get a contradiction. ■

3.4 Theorem

If A and B are subsets of a fuzzy topological space (X,T) and $\mu_A \leq \mu_B \leq \overline{\mu_A}$, if A is fuzzy strong connected subset of X then B is also a fuzzy strong connected.

Proof. Let B is not fuzzy strong connected, then there exist two non zero fuzzy closed sets f/B and k/B such that

$$f/B + k/B \leq 1 \tag{1}$$

If $f/A = 0$ then $f + \mu_A \leq 1$ and this implies

$$f + \mu_A \leq f + \mu_B \leq f + \overline{\mu_A} \tag{2}$$

then $f + \mu_B \leq 1$, thus $f/B = 0$ contradiction, there for $f/A \neq 0$, similarly we can show that $k/A \neq 0$. By (1) with the relation $\mu_A \leq \mu_B$ imply

$$f/A + k/A \leq 1$$

so A is not fuzzy strong connected which is contradiction also. ■

3.5 Theorem

A fuzzy topological space (X,T) is locally fuzzy connected iff every open fuzzy subset of X is fuzzy connected.

Proof. let A be a subspace of X and let η be a fuzzy open set in X. To prove A is fuzzy connected, let x_{α}^{a} be a fuzzy point in A and let η/A be a fuzzy open set in A containing x_{α}^{a} , it must prove that there exist a connected fuzzy open set μ/A in A such that

$$x_{\alpha}^{a} \leq \mu/A \leq \eta/A$$

clearly, the fuzzy point x_{α}in X is contained in η. Since X is locally fuzzy connected, then there exist an open fuzzy connected μ such that

$$x_{\alpha} \leq \mu \leq \eta \text{ and } \mu \leq \eta \wedge \chi_{A}$$

.It is easy prove that

$$x_{\alpha}^{a} \leq \mu/A \leq \eta/A,$$

if μ/A is not fuzzy connected, there exist a proper fuzzy clopen λ/A in μ/A (λ is proper fuzzy clopen in μ) This is contradiction that μ is fuzzy connected and then A is fuzzy connected. ■

In the same way we can prove the case of (X,T) is strong connected space and super connected space.

3.6 Theorem

Let X be a locally super connected and Y be a fuzzy topological space, let F be a fuzzy continuous from X onto Y, then Y is locally super connected.

Proof. Let y_{λ}be a fuzzy point of Y, to prove Y is locally fuzzy super connected i.e for every open set μ in Y containing y_{λ}($y_{\lambda} \leq \mu$) there exist a super connected fuzzy open set η such that $y_{\lambda} \leq \acute{\eta} \leq \mu$. Let $F: X \longrightarrow Y$ be a fuzzy continuous, then there exist a fuzzy point $x_{\delta}$of X such that $F(x_{\delta}) = y_{\lambda}$, $F^{-1}(\mu\ )$ is fuzzy open set in X then

$$F^{-1}(\mu\ )(x_{\delta}) = \mu(F(x_{\delta}) = \mu\ (y_{\lambda}),\ F(x_{1}) \leq \mu$$

thus $x_{\delta} \leq F\ (\mu)$, Since X is locally fuzzy super connected there exist a fuzzy super connected open set η such that

$$x_{\delta} \leq \eta \leq F^{-1}(\mu)$$

then

$$F(x_{\delta}) \leq F(\eta) \leq \mu$$

and then $F(\eta\ )$ is super connected fuzzy.$[7, \text{Th.}(6.5)]$. ■

In the same way we can prove the case for locally fuzzy strong connected space.

4 Other types of connectedness

In this section we proved that in a T_1-fuzzy compact space the notions c-zero dimensional, strong c-zero dimensional and totally c_i-connected are equivalent.

4.1 Definition

let A be a subspace of a fuzzy to0000pological space (X, T) and let $\{u_s\}_{s\in S}$ be a family of fuzzy open subsets of X such that $A \leq \vee_{s\in S} u_s$. If A is fuzzy compact then there exist a finite set $\{s_1, s_2, ...s_k\}$ such that $A \leq \vee_{i=1}^{k} u_{s_i}$.

4.2 Theorem

In a fuzzy topological space (X,T) a fuzzy quasi-component {Q} is less than the fuzzy quasi-component {C} for every point x_1.

Proof. Let x_1 be a fuzzy point in (X,T), suppose $C \nleq Q$, let μ be any fuzzy clopen subset of X contain x_1, then we consider $C \wedge \mu$ and $C - \mu$. It is clear that $C \wedge \mu \neq 0$

$$(C - \mu)(x) = \{ \begin{array}{ccc} C(x) & if & C(x) > \mu(x) \\ 0 & & \text{otherwise} \end{array}$$

If $C - \mu = C$ this mean

$$(C \wedge \mu)(x) + (C - \mu\ )(x) = 1c$$

Contradiction, it must be $C - \mu\ = 0$, then $C \leq Q$ since μ is arbitrary, thus $C \leq Q$. ■

4.3 Lemma

Let μ be a fuzzy open subset of a topological space (X, T). If a family $\{Fs\}_{s\in S}$ of closed subset of X contains at least one fuzzy compact set, in particular if X is fuzzy compact and if $\wedge_{s\in S} Fs < \mu$, there exists a finite set $\{s_1, s_2, ...s_k\}$ such that $\wedge_{i=1}^{k} Fs_i < \mu$.

Proof. let μ be a fuzzy open set, then $1 - \mu = \mu^c$ is fuzzy closed

$$(\wedge_{s\in S} Fs < \mu)^c = \vee_{s\in S} F_s^c > \mu^c = 1 - \mu$$

which is fuzzy compact [every fuzzy closed of fuzzy compact is fuzzy compact].

$$1 - \mu < \vee_{s\in S} F_s^c$$

then

$$1 - \mu < \vee_{i=1}^{k} F_{s_i}^c$$

and then

$$\wedge_{i=1}^{k} Fs_i < \mu.$$

■

4.4 Theorem

Let (X, T) be a T_1- fuzzy compact space then the following are equivalent.

1- (X, T) is $c - zero$ dimensional.

2-(X, T) is strong $c - zero$ dimensional.

3-(X, T) is totally $c_i - connected, i = 1, 2, 3, 4$.

Proof. (1) $\Rightarrow$ (2) it is clear that every fuzzy compact is fuzzy Lindeloff topological space, then by [Th.(4.5), 4], (X,T) is strongly $c - zero$ dimensional.

(2) $\Rightarrow$ (1), by [Th.(4.4), 4].

(1) $\Rightarrow$ (3), by [Th.(4.3), 4], (X, T) is totally $c_i - connected, i = 1, 2, 3, 4$.

(3) $\Rightarrow$ (1), it must prove that if (X, T) is fuzzy compact totally $c_i - connected$, then it is $c - zero$ dimensional. let x_1 be a crisp fuzzy point in X and let μ be a fuzzy open set containing x_1, to prove that there exists a crisp clopen fuzzy set δ in X such that

$$x_1 \leq \delta \leq \mu$$

Let

$$\mu^* = \{\mu : \mu \text{ is fuzzy clopen and } x_1 \leq \mu\}$$

is fuzzy quasi-component of x_1, let U be a neighborhood of x_1, $\mu^* < U$ then by [Lemma (4.3)] and the fuzzy compactness of (X, T),

$$\mu^* = \mu_1 \wedge \ \mu_2 \wedge ..... \wedge \mu_k < U$$

and then we get

$$x_1 \leq \mu^* < U.$$

■

5 Conclusion

We studied some fuzzy topological spaces such fuzzy super connected, fuzzy strongly connected, c-zero dimensional, total disconnected and strongly zero connected, we gave more results on these spaces and prove that locally fuzzy connectedness is a good extension of locally connectedness. Also, it is proved that in a T_1-fuzzy compact space the notions c-zero dimensional, Strong c-zero dimensional and totally c_i-connected are equivalents.

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