On some kinds of fuzzy connected space

by

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Abstract

In this paper new results in fuzzy connected space are obtaind and prove that locally fuzzy connectedness is good extension of locally connectedness also it is proved that in a $T_1$-fuzzy compact space the notations c-zero dimensional, Strong c-zero dimensional and Totally $c_i$-connected are equivalents.

Keywords: Fuzzy connected space, fuzzy strong connected, fuzzy super connected, c-zero dimensional, strong c-dimensional and totally $c_i$-connected.

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1 Introduction

In 1965 Zadeh in his classical paper generalized characteristic functions to fuzzy sets. Chang in 1968 introduced the topological structure fuzzy sets in a given set. Pu and Liu defined connectedness by using the concept fuzzy closed set. Lowen also defined an extension of a connectedness in a restricted family of fuzzy topologies. Fatteh and Bassam studied further the notion of fuzzy super connected and fuzzy strongly connected spaces. However they defined connectedness only for a crisp set of a fuzzy topological space. Ajmal and Kohli extend the notations of connectedness to an arbitrary fuzzy set, and defined the notions of c-zero dimensional, total disconnectedness and strongly zero connectedness.

In this paper we give more results on these spaces and prove that locally fuzzy connectedness is a good extension of locally connectedness. Also, it is proved that in a $T_1$-fuzzy compact space the notions c-zero dimensional, Strong c-zero dimensional and totally $c_i$-connected are equivalents.
2 Basic definitions

The following definitions have been used to obtain the results and properties developed in this paper.

Definition 1 [7] A fuzzy topological space $X$ is said to be fuzzy connected if it has no proper fuzzy clopen set. (A fuzzy set $\lambda$ in $X$ is proper if $\lambda \neq 0$ and $\lambda \neq 1$).

Definition 2 [7] A fuzzy topological space $X$ is said to be fuzzy super-connected if $X$ does not have non-zero fuzzy open sets $\lambda$ and $\mu$ such that $\lambda + \mu \leq 1$.

Definition 3 [7] A fuzzy topological space $X$ is said to be fuzzy strongly connected if it has no non-zero fuzzy closed sets $f$ and $k$ such that $f + k \leq 1$.

Definition 4 [1] A fuzzy topological space $X$ is said to be c-zero dimensional if every crisp fuzzy point $x_1$ in $X$ and every fuzzy open set $\mu$ containing $x_1$ there exists a crisp clopen fuzzy set $\delta$ in $X$ such that $x_1 \leq \delta \leq \mu$.

Definition 5 [1] A fuzzy topological space $X$ is said to be totally $c_i$-disconnected ($i = 1, 2, 3, 4$) if the support of every non-zero $c_i$-connected fuzzy set in $X$ is a singleton.

Definition 6 [1] A fuzzy topological space $X$ is said to be strongly $c$-zero dimensional if it is not $c$-connected between any pair of its disjoint fuzzy closed sets.

Definition 7 [2] A fuzzy topological space $X$ is said to be fuzzy locally connected at a fuzzy point $x_\alpha$ in $X$ if for every fuzzy open set $\mu$ in $X$ containing $x_\alpha$, there exists a connected fuzzy open set $\delta$ in $X$ such that $x_1 \leq \delta \leq \mu$.

Definition 8 A fuzzy topological space $(X,T)$ is said to be locally fuzzy super connected (locally fuzzy strong connected) at a fuzzy point $x_\alpha$ in $X$ if for every fuzzy open set $\mu$ in $X$ containing $x_\alpha$, there exist a fuzzy super connected (fuzzy strong connected) open set $\eta$ in $X$ such that $x_\alpha \leq \eta \leq \mu$.

Definition 9 A fuzzy quasi-component of a fuzzy point $x_1$ in a fuzzy topological space $(X,T)$ is the smallest fuzzy clopen subset of $X$ contain $x_1$ and denoted by $Q$.

Definition 10 A fuzzy path component of a fuzzy point $x_1$ in a fuzzy topological space $(X,T)$ is the maximal fuzzy path connected in $(X,T)$ contain $x_1$ and denoted by $C$.

3 Fuzzy connectedness and its stronger forms

In this section we study some stronger forms of connectedness such as fuzzy super connected, totally $c_i$-disconnected and fuzzy strongly connected introduced by Fatteh and Bassam and we proved that locally fuzzy connectedness is a good extension of locally connectedness also we get some additional results and properties for these spaces.
3.1 Theorem

A topological space \((X, \tau)\) is locally connected if and only if \((X, \omega(\tau))\) is locally connected.

Proof. Let \(\mu\) be a fuzzy open set in \(\omega(\tau)\) containing a fuzzy point \(x_\alpha\), since \(\mu\) is lower semicontinuous function, then by local connectedness of \((X, \tau)\) there exists an open connected set \(U\) in \(X\) containing \(x\) and contained in the support of \(\mu\), i.e \((x \in U \subset \text{Supp } \mu)\). Now \(\chi_U\) is the characteristic function of \(U\) and it is a lower semicontinuous, then \(\chi_U \wedge \mu\) is fuzzy open set in \(\omega(\tau)\). We claim \(\delta = \chi_U \wedge \mu\) is fuzzy connected set containing \(x_\alpha\), if not then by [8, Th.(3.1),] there exists a non zero lower semicontinuous functions \(\mu_1, \mu_2\) in \(\omega(\tau)\) such that

\[ \mu_1 | \delta + \mu_2 | \delta = 1 \]

Now \(\text{Supp } \delta = U\) and \(\text{Supp } \mu_1, \text{Supp } \mu_2\) are open sets in \(\tau\) such that

\[ U \subset \text{Supp } \mu_1 \cup \text{Supp } \mu_2 \]

then,

\[ U \cap \text{Supp } \mu_1 \neq \Phi \]

and

\[ U \cap \text{Supp } \mu_2 \neq \Phi \]

then

\[ (U \cap \text{Supp } \mu_1) \cup (U \cap \text{Supp } \mu_2) = U \cap (\text{Supp } \mu_1 \cup \text{Supp } \mu_2) = U \]

is not connected. Conversely let \(U\) be an open set in \(\tau\) containing \(x, x_\alpha \in \chi_U\), \((\chi_u\) is the characteristic function of \(U\), \(\chi_u\) is fuzzy open set in \(\omega(\tau)\). By fuzzy connectedness of \((X, \omega(\tau))\) there exists a fuzzy open connected set \(\mu\) in \(\omega(\tau)\) such that.

\[ x_\alpha \leq \mu \leq \chi_U \]

Claim that \(\text{Supp } \mu\) is connected \((x \in \text{Supp } \mu \subset U)\), if not there exists two non empty open sets \(G_1, G_2 \in \tau\) such that

\[ \text{Supp } \mu = G_1 \cup G_2 \text{ and } G_1 \cap G_2 = \Phi \]

it is clear that

\[ \chi_{G_1} + \chi_{G_2} = 1_\mu \]

which is contradiction, because \(\mu\) is fuzzy connected. \(\blacksquare\)
3.2 Theorem

If G is a subset of a fuzzy topological space (X,T) such that \( \mu_G \) (\( \mu_G \) is the characteristic function of a subset \( G \) of \( X \)) is fuzzy open in \( X \), then if \( X \) is super connected space implies \( G \) is fuzzy super connected space.

Proof. Suppose that \( G \) is not fuzzy super connected space then by [7, Th.(6.1),] exists fuzzy open sets \( \lambda_1, \lambda_2 \) in \( X \) such that \( \lambda_1/G \neq 0, \lambda_2/G \neq 0 \) and \( \lambda_1/G + \lambda_2/G \leq 1 \) there for \( \lambda_1 \land \mu_G + \lambda_2 \land \mu_G \leq 1 \).

Then \( X \) is not fuzzy super connected space and we get contradiction. ■

3.3 Theorem

If \( A \) and \( B \) are fuzzy strong connected subsets of a fuzzy topological space (X,T) and \( \mu_B/A \neq 0 \) or \( \mu_A/B \neq 0 \), then \( A \lor B \) is a fuzzy strong connected subset of \( X \) where \( \mu_A, \mu_B \) are the characteristic function of a subset \( A \) and \( B \) respectively.

Proof. Suppose \( Y = A \lor B \) is not strong fuzzy connected subset of \( X \). Then there exist fuzzy closed sets \( \delta \) and \( \lambda \) such that \( \delta/Y \neq 0 \) and \( \lambda/Y \neq 0 \) and \( \delta/Y + \lambda/Y \leq 1 \). Since \( A \) is fuzzy strong connected subset of \( X \), then either \( \delta/A = 0 \) or \( \lambda/A = 0 \). Without loss the generality assume that \( \delta/A = 0 \) in this case since \( B \) is also fuzzy strong connected we have

\[ \delta/A = 0, \lambda/A \neq 0, \delta/B \neq 0, \lambda/B = 0 \]

there for

\[ \lambda/A + \mu_B/A \leq 1 \]  \( (1) \)

If \( \mu_B/A \neq 0 \), then \( \lambda/A \neq 0 \) with \( (1) \) imply that \( A \) is not fuzzy connected subset of \( X \). In the same way if \( \mu_B/B \neq 0 \) then \( \delta/B \neq 0 \) and \( \lambda/B + \mu_B/B \leq 1 \) imply that \( B \) is not a fuzzy strong connected subset of \( X \), we get a contradiction. ■

3.4 Theorem

If \( A \) and \( B \) are subsets of a fuzzy topological space (X,T) and \( \mu_A \leq \mu_B \leq \mu_A \), if \( A \) is fuzzy strong connected subset of \( X \) then \( B \) is also a fuzzy strong connected.

Proof. Let \( B \) is not fuzzy strong connected, then there exist two non zero fuzzy closed sets \( f/B \) and \( k/B \) such that

\[ f/B + k/B \leq 1 \]  \( (1) \)

If \( f/A = 0 \) then \( f + \mu_A \leq 1 \) and this implies

\[ f + \mu_A \leq f + \mu_B \leq f + \mu_A \]  \( (2) \)

then \( f + \mu_B \leq 1 \), thus \( f/B = 0 \) contradiction, there for \( f/A \neq 0 \), similarly we can show that \( k/A \neq 0 \). By \( (1) \) with the relation \( \mu_A \leq \mu_B \) imply

\[ f/A + k/A \leq 1 \]

so \( A \) is not fuzzy strong connected which is contradiction also. ■
3.5 Theorem
A fuzzy topological space \((X, T)\) is locally fuzzy connected if and only if every open fuzzy subset of \(X\) is fuzzy connected.

**Proof.** Let \(A\) be a subspace of \(X\) and let \(\eta\) be a fuzzy open set in \(X\). To prove \(A\) is fuzzy connected, let \(x_\alpha^a\) be a fuzzy point in \(A\) and let \(\eta/A\) be a fuzzy open set in \(A\) containing \(x_\alpha^a\), it must prove that there exist a connected fuzzy open set \(\mu/A\) in \(A\) such that

\[
x_\alpha^a \leq \mu/A \leq \eta/A
\]

Clearly, the fuzzy point \(x_\alpha^a\) in \(X\) is contained in \(\eta\). Since \(X\) is locally fuzzy connected, then there exist an open fuzzy connected \(\mu\) such that

\[
x_\alpha \leq \mu \leq \eta \text{ and } \mu \leq \eta \wedge \chi_A
\]

It is easy prove that

\[
x_\alpha^a \leq \mu/A \leq \eta/A,
\]

if \(\mu/A\) is not fuzzy connected, there exist a proper fuzzy clopen \(\lambda/A\) in \(\mu/A\) (\(\lambda\) is proper fuzzy clopen in \(\mu\)) This is contradiction that \(\mu\) is fuzzy connected and then \(A\) is fuzzy connected. □

In the same way we can prove the case for locally fuzzy strong connected space.

3.6 Theorem
Let \(X\) be a locally super connected and \(Y\) be a fuzzy topological space, let \(F\) be a fuzzy continuous from \(X\) onto \(Y\), then \(Y\) is locally super connected.

**Proof.** Let \(y_\lambda\) be a fuzzy point of \(Y\), to prove \(Y\) is locally fuzzy super connected i.e for every open set \(\mu\) in \(Y\) containing \(y_\lambda\) \(y_\lambda \leq \eta\leq \mu\) there exist a super connected fuzzy open set \(\eta\) such that \(y_\lambda \leq \eta\leq \mu\). Let \(F : X \longrightarrow Y\) be a fuzzy continuous, then there exist a fuzzy point \(x_\delta\) of \(X\) such that \(F(x_\delta) = y_\lambda\), \(F^{-1}(\mu)\) is fuzzy open set in \(X\) then

\[
F^{-1}(\mu)(x_\delta) = \mu(F(x_\delta) = \mu_{y_\lambda}, F(x_\delta) \leq \mu
\]

Thus \(x_\delta \leq F(\mu)\), Since \(X\) is locally fuzzy super connected there exist a fuzzy super connected open set \(\eta\) such that

\[
x_\delta \leq \eta \leq F^{-1}(\mu)
\]

then

\[
F(x_\delta) \leq F(\eta) \leq \mu
\]

and then \(F(\eta)\) is super connected fuzzy. [7, Th.(6.5)] □

In the same way we can prove the case for locally fuzzy strong connected space.
4 Other types of connectedness

In this section we proved that in a $T_1$-fuzzy compact space the notions $c$-zero dimensional, strong $c$-zero dimensional and totally $c_i$-connected are equivalent.

4.1 Definition

Let $A$ be a subspace of a fuzzy topological space $(X, T)$ and let $\{u_s\}_{s \in S}$ be a family of fuzzy open subsets of $X$ such that $A \leq \bigvee_{s \in S} u_s$. If $A$ is fuzzy compact then there exist a finite set $\{s_1, s_2, \ldots, s_k\}$ such that $A \leq \bigvee_{i=1}^{k} u_{s_i}$.

4.2 Theorem

In a fuzzy topological space $(X, T)$ a fuzzy quasi-component $\{Q\}$ is less than the fuzzy quasi-component $\{C\}$ for every point $x_1$.

Proof. Let $x_1$ be a fuzzy point in $(X, T)$, suppose $C \nleq Q$, let $\mu$ be any fuzzy clopen subset of $X$ contain $x_1$, then we consider $C \wedge \mu$ and $C - \mu$. It is clear that $C \wedge \mu \neq 0$

$$(C - \mu)(x) = \begin{cases} C(x) & \text{if } C(x) > \mu(x) \\ 0 & \text{otherwise} \end{cases}$$

If $C - \mu = C$ this mean

$$(C \wedge \mu)(x) + (C - \mu)(x) = 1$$

Contradiction, it must be $C - \mu = 0$, then $C \leq Q$ since $\mu$ is arbitrary, thus $C \leq Q$.

4.3 Lemma

Let $\mu$ be a fuzzy open subset of a topological space $(X, T)$. If a family $\{F_s\}_{s \in S}$ of closed subset of $X$ contains at least one fuzzy compact set, in particular if $X$ is fuzzy compact and if $\wedge_{s \in S} F_s < \mu$, there exists a finite set $\{s_1, s_2, \ldots, s_k\}$ such that $\wedge_{i=1}^{k} F_{s_i} < \mu$.

Proof. Let $\mu$ be a fuzzy open subset, then $1 - \mu = \mu^c$ is fuzzy closed

$$(\wedge_{s \in S} F_s < \mu)^c = \vee_{s \in S} F_s^c > \mu^c = 1 - \mu$$

which is fuzzy compact [every fuzzy closed of fuzzy compact is fuzzy compact].

$$1 - \mu < \vee_{s \in S} F_s^c$$

then

$$1 - \mu < \vee_{i=1}^{k} F_{s_i}^c$$

and then

$$\wedge_{i=1}^{k} F_{s_i} < \mu.$$
4.4 Theorem

Let \((X, T)\) be a \(T_1\)-fuzzy compact space then the following are equivalent.

1- \((X, T)\) is \(c\)–zero dimensional.
2- \((X, T)\) is strong \(c\)–zero dimensional.
3- \((X, T)\) is totally \(c_i\)–connected, \(i = 1, 2, 3, 4\).

**Proof.** (1) \(\Rightarrow\) (2) it is clear that every fuzzy compact is fuzzy Lindeloff topological space, then by [Th.(4.5), 4], \((X,T)\) is strongly \(c\)–zero dimensional.

(2) \(\Rightarrow\) (1), by [Th.(4.4), 4].

(1) \(\Rightarrow\) (3), by [Th.(4.3), 4], \((X,T)\) is totally \(c_i\)–connected, \(i = 1, 2, 3, 4\).

(3) \(\Rightarrow\) (1), it must prove that if \((X,T)\) is fuzzy compact totally \(c_i\)–connected, then it is \(c\)–zero dimensional. let \(x_1\) be a crisp fuzzy point in \(X\) and let \(\mu\) be a fuzzy open set containing \(x_1\), to prove that there exists a crisp clopen fuzzy set \(\delta\) in \(X\) such that

\[x_1 \leq \delta \leq \mu\]

Let

\[\mu^* = \{\mu : \mu \text{ is fuzzy clopen and } x_1 \leq \mu\}\]

is fuzzy quasi-component of \(x_1\), let \(U\) be a neighborhood of \(x_1\), \(\mu^* \leq U\) then by [Lemma (4.3)] and the fuzzy compactness of \((X,T)\),

\[\mu^* = \mu_1 \land \mu_2 \land \ldots \land \mu_k < U\]

and then we get

\[x_1 \leq \mu^* < U\]

\[\blacksquare\]

5 Conclusion

We studied some fuzzy topological spaces such fuzzy super connected, fuzzy strongly connected, \(c\)-zero dimensional, total disconnected and strongly zero connected, we gave more results on these spaces and prove that locally fuzzy connectedness is a good extension of locally connectedness. Also, it is proved that in a \(T_1\)-fuzzy compact space the notions \(c\)-zero dimensional, Strong \(c\)-zero dimensional and totally \(c_i\)-connected are equivalents.

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References


