# COMPUTATIONAL STUDY OF PARTIAL DIFFERENTIAL EQUATIONS WITH COMPRESSIBLE TWO–PHASE FLOW APPLICATION

DIA ZEIDAN*
Department of Mathematics
Al–Zaytoonah University of Jordan
P.O. Box 130, Amman 11733, Jordan

**Abstract**

The one–dimensional differential equations for the conservation mixture mass and mixture momentum coupled by a relative velocity conservation equation have been solved for an isentropic mixture of gas and liquid two–phase flow. Under certain restrictions the resulting system of partial differential equations is hyperbolic and allow discontinuous solutions. These equations have been solved by the TVD SLIC scheme with good accuracy in a simple way. To illustrate the character of the solution of the model, results are presented for an isentropic gas–liquid mixture two–phase flow Riemann problem. The solution is validated against existing exact results in conjunction with the SLIC scheme.

*Keywords:*
*Simplified Two–phase Flow Model; Hyperbolic Partial Differential Equations; Finite Volume Numerical Methods*

*Contribution to the Proceeding of the* $2^{nd}$ *International Conference on Information Technology 2005*

## 1 Introduction

The field of partial differential equations (PDE's) is a major link between mathematics and its applications. Focusing only on a particular field partial differential equations such as fluid

* *E–mail address:* dia@alzaytoonah.edu.jo

flow phenomena, one may develop a mathematical model as a starting point for a numerical analysis along with initial conditions, boundary conditions and closure relations. This requires a good understanding of the basic theory of partial differential equations and the processes they represent [1, 2, 7]. For instance, one must be sure weather the boundary conditions are physically correct and that the problem to be solved is well–posed. When solving partial differential equations numerically one should be in a position of knowing what to expect from the numerical method. The physical and mathematical interpretation of the solution will be always an essential part of the computational work. Focusing in a typical engineering context, a substance exists in one of three physical phases or states (excluding the plasma state): a gas, a liquid, or a solid. Multiphase flow is the simultaneous flow of several phases. Two–phase flow is the simplest case of multiphase flow.

Two–phase flow computational fluid dynamics (CFD) is one of the most challenging research fields in applied and computational mathematics. A common theory of two–phase flow modelling is to start with a single continuous description of each phase given by the Navier–Stokes equations [3]. However, such equations are generally not hyperbolic and possess complex characteristics [4]. This creates severe technical difficulties as the initial–value problem is then ill–posed, and serious numerical problems that may render accurate computation impossible. Complex characteristics arise as a result of two–phase flow averaging, and the consequent omission of important physical terms. With care, however, these effects may be reintroduced into the equations. The difficulties associated with two–phase flow models can be significantly reduced by considering homogenous or diffusion models in which the relative velocity between phases is expressed by a constitutive equation [10].

The computation presented in this paper is an extension of previous work [8, 9]. We consider the one–dimensional simplified mixture differential equations two–fluid model for an isentropic mixture two–phase flow coupled by a relative velocity conservation equation. The simplified mixture differential equations two–fluid model results in a mathematically conservative form well–posed initial–value problem provided that the relative velocity between the two phases is much lower than the speed of sound of the two phases mixture [10]. Moreover, the equations well suited for using numerical methods specifically developed for high speed single–phase flows such as Godunov–type methods [5] and to solve the Riemann problem for the two–phase flow configurations.

We propose to use Godunov methods of centred–type such as the total variation diminishing (TVD) slope limiter centred (SLIC) scheme [6] which have the property that it is a second–order accurate and oscillation–free across discontinuities. Centred methods are methods do not require the explicit solution of the Riemann Problem, i.e. they are not based by the wave propagation direction. The SLIC scheme will give insight into the equations for the simplified mixture model and resolve the Riemann problem as we shall see through the numerical results presented in this paper. The numerical results for an isentropic gas–liquid mixture two–phase flow Riemann problem closely match the results obtained by the analytical solution of the Riemann problem [10].

For comparison we also show numerical results obtained by the Lax–Friedrichs scheme. Following the ideas of the previous work, we recast the equations into a form more suited for analysis and numerical computation.

## 2 Computational Model–A Simplified Two–Fluid Model

The simplest model for a substance which consists of inviscid phases contains the differential forms of the conservation laws of mass and momentum with no viscosity effect and no mass transfer between phases. Thus no right–hand terms for the equations exist. The model presented here contains of three partial differential equations of mixture mass, mixture momentum and relative velocity between the two phases. These equations describe an isentropic two–phase flow such as a mixture of gas and liquid. Therefore, single variables describing the state of the gas–liquid medium and single pressure hypothesis are introduced. The set of variables required include the mixture mass density $\rho$ and the mass gas concentration $c$

$$\rho = \alpha\rho_2 + (1-\alpha)\rho_1, \tag{1}$$

and

$$c = \frac{\alpha\rho_2}{\rho}, \tag{2}$$

the mixture velocity

$$u = \frac{\alpha\rho_2 + (1-\alpha)\rho_1}{\rho}, \tag{3}$$

and the generalised equation of state in a mixture formulation

$$\rho = \rho(\mathcal{P}), \tag{4}$$

with the volume fractions related by

$$(\alpha - 1) + \alpha = 1. \tag{5}$$

The variable responsible for the momentum exchange is the velocity difference between the mixture components

$$u_r = u_2 - u_1. \tag{6}$$

For this set of variables, the standard one–dimensional equations for a two phase medium can be written in a form which contains the conservation laws for mixture variables and relative velocity. These are given as the following set of equations [10]:

$$\big[\rho\big]_t + \big[\rho u\big]_x = 0, \tag{7}$$

$$\big[\rho u\big]_t + \big[\rho u^2 + \rho c(1-c)u_r^2 + \mathcal{P}\big]_x = 0, \tag{8}$$

$$\big[u_r\big]_t + \big[u u_r + (0.5-c)u_r^2 + \psi(\mathcal{P})\big]_x = 0. \tag{9}$$

This conservation form of the isentropic mixture two–phase flow is with respect to the morphology of the simplified two–fluid model. The model will be useful to study the gas–liquid mixture two–phase flow Riemann problem and to define a discontinuous solution such as contacts and shock waves. We refer to [10] for further details on the theoretical and numerical analysis of the simplified two–fluid model.

## 3 Solution Technique

For two–phase flows it is possible to make a certain amount of progress analytically by following the ideas of Zeidan [10] and by looking for Riemann problem solutions. In [10] I have looked at the flow of two fluids and studied Riemann problem two–phase flows with a velocity difference. It was possible to construct an exact and approximate Riemann solvers, and to develop Godunov–type methods for hyperbolic conservative models.

Instead of looking for analytical solutions or Godunov methods of upwind–type it is possible to solve the conservation equations for two–phase flows by Godunov methods of centred–type for which the solution of the Riemann problem is fully numerical. As for two–phase flows, the Riemann problem for the simplified two–fluid model is the initial–value problem for the conservation law

$$\partial_t \mathbb{U} + \partial_x \mathbb{F}(\mathbb{U}) = 0, \qquad x \in \mathbb{R}, \quad t > 0, \tag{10}$$

with the initial conditions (IC)

$$\mathbb{U}(x,0) = \begin{cases} \mathbb{U}_{\mathcal{L}} & \text{if} \quad x < 0, \\ \mathbb{U}_{\mathcal{R}} & \text{if} \quad x > 0, \end{cases} \tag{11}$$

where $\mathcal{L}$ and $\mathcal{R}$ denote the left and right states respectively, $\mathbb{U}$ and $\mathbb{F}(\mathbb{U})$ are given by

$$\mathbb{U} = \begin{pmatrix} \rho \\ \rho u \\ u_r \end{pmatrix}, \qquad \mathbb{F}(\mathbb{U}) = \begin{pmatrix} \rho u \\ \rho u^2 + \rho\ c\ (1-c)\ u_r^2 + \mathcal{P} \\ u u_r + (0.5 - c)\ u_r^2 + \psi(\mathcal{P}) \end{pmatrix}. \tag{12}$$

The initial conditions (11) consists of two constant states $\mathbb{U}_{\mathcal{L}}$ and $\mathbb{U}_{\mathcal{R}}$ with

$$\mathbb{U}_{\mathcal{L}} = \begin{pmatrix} \rho_{\mathcal{L}} \\ \rho_{\mathcal{L}} u_{\mathcal{L}} \\ u_{r\mathcal{L}} \end{pmatrix}, \qquad \mathbb{U}_{\mathcal{R}} = \begin{pmatrix} \rho_{\mathcal{R}} \\ \rho_{\mathcal{R}} u_{\mathcal{R}} \\ u_{r\mathcal{R}} \end{pmatrix}, \tag{13}$$

and represent conditions at time $t = 0$, to the left of $x = 0$ and to the right of $x = 0$, respectively. The simplified two–fluid model equations have been solved using the TVD SLIC scheme to remove spurious numerical oscillations at discontinuities. The SLIC scheme advances the solution using the conservative formula

$$\mathbb{U}_i^{n+1} = \mathbb{U}_i^n - \frac{\Delta t}{\Delta x}\left[\mathbb{F}_{i+\frac{1}{2}} - \mathbb{F}_{i-\frac{1}{2}}\right], \tag{14}$$

where $\mathbb{F}_{i+\frac{1}{2}}$ represent the numerical flux corresponding to cell $i$ and $i+1$, and $\Delta t$ satisfies the condition

$$\Delta t = C_{CFL} \frac{\Delta x}{S^{(n)}_{max}}, \tag{15}$$

where $C_{CFL}$ is the CFL (Courant–Friedrichs–Lewy) number, is chosen within the interval $(0, 1]$, and $S^{(n)}_{max}$ is the maximum wave speed present at time level $n$ chosen to be

$$\mathbb{S}^{(n)}_{max} = \max_i \left\{ |u^n_i| + a^n_{mi} + \frac{\rho^n_i}{\rho^n_{21i}} c^n_i (1 - c^n_i) u^n_{ri} \right\}. \tag{16}$$

We shall not deal upon the detailed description of the scheme. For a clear and detailed presentation of the scheme, the reader is particularly recalled to [10] for two–phase flow and [6] for single–phase case.

## 4 Numerical Results

To illustrate the effectiveness of the computational study of partial differential equations for two–phase flows we solved an isentropic gas–liquid mixture two–phase flow Riemann problem. Results are presented to illustrate typical wave patterns resulting from the solution of the Riemann problem. These characterised by a left shock wave associated with the left eigenvalue and a right shock wave associated with the right eigenvalue separated by a contact discontinuity. The aim of this test problem is to assess the ability of numerical methods to resolve shock waves in general and shock waves in two–phase flow in particular. Generally, the ability to resolve shock waves with high–resolution and without spurious oscillation in the vicinity of discontinuity depends on the availability of robust and accurate numerical methods. The TVD SLIC scheme seems to be quite satisfactory with these requirements. The numerical solution to the shock waves test problem is obtained by running a FORTRAN 90 program in the spatial domain $-10 \leq x \leq 10$ with a computational mesh of 100 cells. Transmissive boundary conditions with a CFL stability coefficient of 0.9 were used together with SUPERBEE limiter. The left and right states for the shock waves test problem are: $\rho_{\mathfrak{L}} = 483.0 = \rho_{\mathfrak{R}}$, $u_{\mathfrak{L}} = 3.345$, $u_{\mathfrak{R}} = 3.0$, $u_{r\mathfrak{L}} = 4.0$, $u_{r\mathfrak{R}} = -3.0$. The numerical results were compared with the analytical solution obtained using an exact Riemann solver [10]. We also compare the results with the Lax–Friedrichs scheme.

Figure 1 shows comparisons between exact (line) and numerical solutions (symbol) at given time obtained by the SLIC scheme. The quantities shown are mixture density $\rho$, mixture velocity $u$, relative velocity $u_r$ and pressure $\mathcal{P}$. For comparison, the proposed shock waves test problem solved with the Lax–Friedrichs scheme, see figure 2. The Lax–Friedrichs scheme has more numerical diffusion than the TVD SLIC scheme, but the results obtained with the two schemes are in good agreement. Clearly, the results of Lax–Friedrichs show their characteristic property of pairing cell values. As seen in figures 1 and 2 the solution of an isentropic mixture two–phase flow Riemann problem consists of three discontinuities: two shock waves and a contact. The

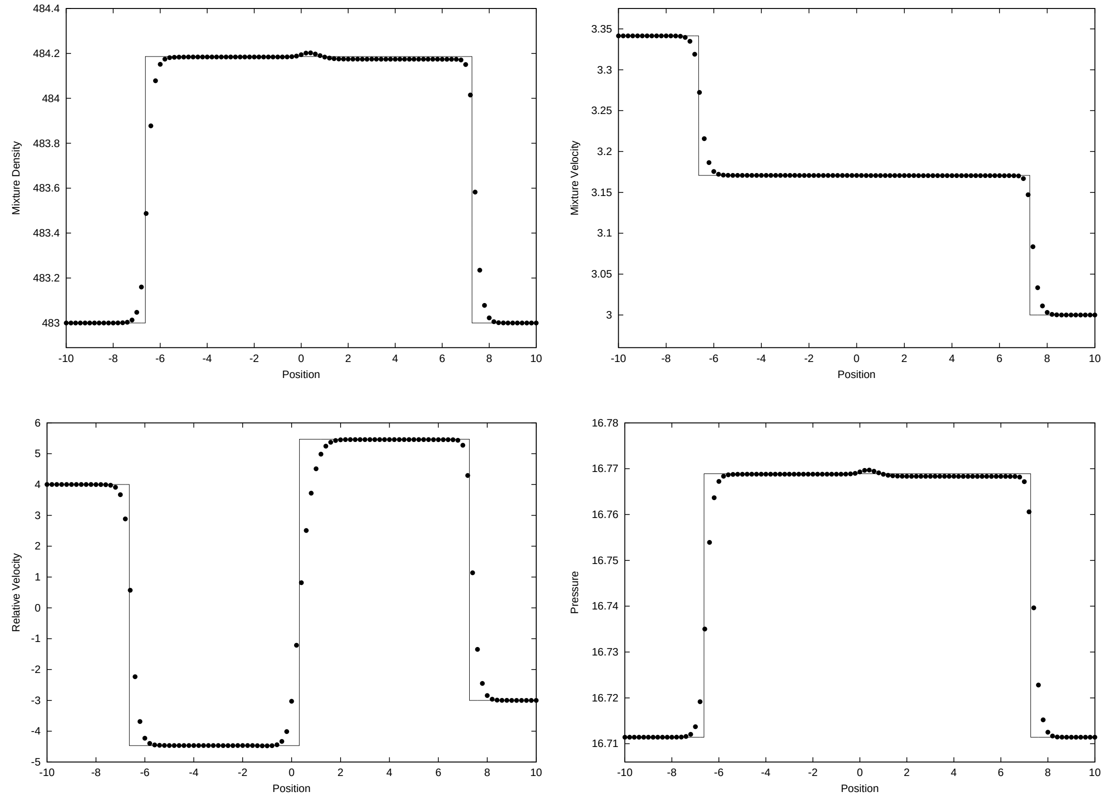


Figure 1: SLIC scheme applied to an isentropic gas–liquid mixture two–phase flow Riemann problem. Numerical (symbol) and exact solutions are compared at time $t = 0.1$ with SUPERBEE limiter.

complete wave system has resulted from the interaction of two strong shock waves propagating in opposite directions.

# 5 Conclusions

This paper has concentrated on homogenous, mixture two–phase flows which are of great importance and interest to the petroleum engineering and safety relevant issues. A unified model of two–phase flows has been considered by solving the one–dimensional equations for conservation of mass and momentum for an isentropic mixture two–phase flow and relative velocity balance law between the two–phases. The model is given by a system of hyperbolic partial differential equations in a conservative form. Although the model is essentially very simple it provides the basis for a useful scientific tool enabling answers to be given to considerable practical problems of great interest to the petroleum engineering and safety relevant issues.

We have presented an extension to two–phase flow calculations of Godunov methods of centred–type such as the SLIC scheme which turned out to be very efficient for two–phase flow calculation

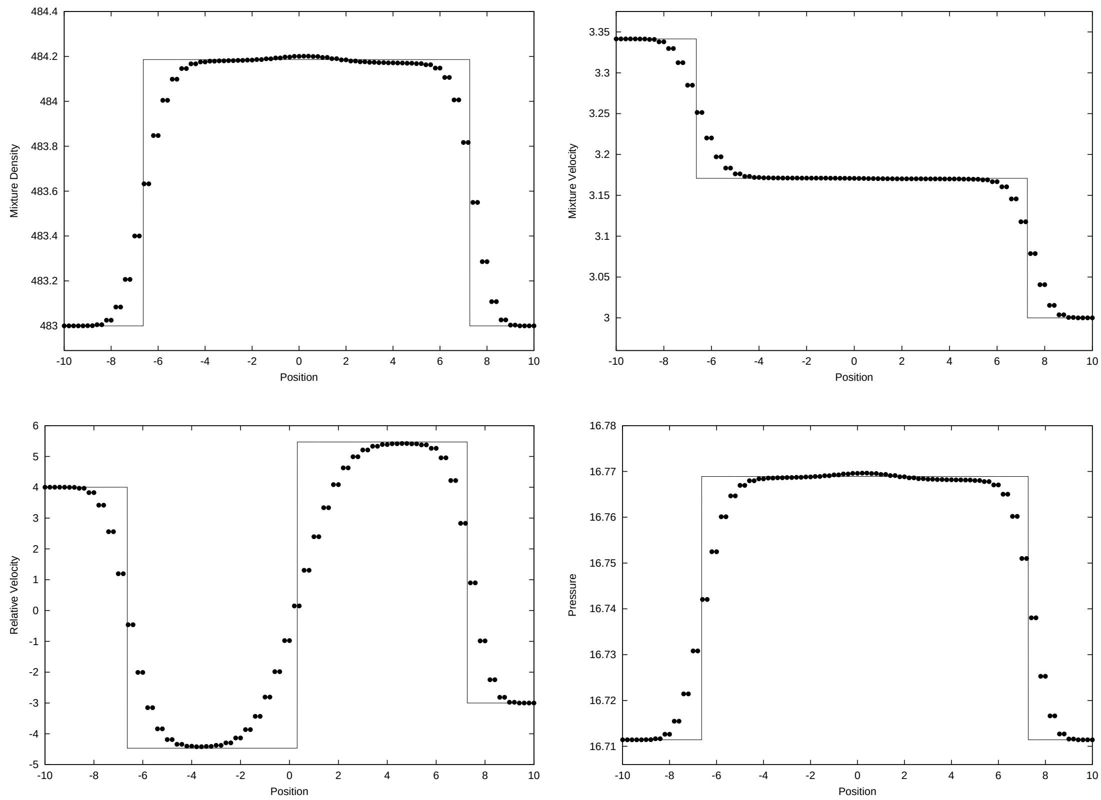


Figure 2: Lax–Friedrichs scheme applied to an isentropic gas–liquid mixture two–phase flow Riemann problem. Numerical (symbol) and exact solutions are compared at time $t = 0.1$ with SUPERBEE limiter.

in one–dimension. The SLIC scheme was successfully tested, showing the accuracy and ability to model flows as shown by the numerical results for the two–phase flow Riemann problem.

## References


[1] Courant R. and Friedrichs K. *Supersonic Flow and Shock Waves.* Springer–Verlag, 1985.

[2] Garabedian P. *Partial Differential Equations.* Wiley. New York, 1964.

[3] Ishii M. *Thermo–Fluid Dynamic Theory of Two–Phase Flow.* Eyrolles. Paris, 1975.

[4] Stewart H. and Wendroff B. Two–Phase Flow: Models and Methods, *J. Comp. Phys.* 1984; **56**: 363–409.

[5] Toro E. F. *Riemann Solvers and Numerical Methods for Fluid Dynamics.* A Practical Introduction. Springer–Verlag, Berlin, 1999.

[6] Toro E. F. and Billett S. Centred TVD Schemes for Hyperbolic Conservation Laws. *IMA. J. Numerical Analysis.* 2000; **20**: 47–79.

[7] Wallis G. *One–Dimensional Two–Phase Flow.* McGraw–Hill. New York, 1969.

[8] Zeidan D., Toro E. F. and Slaouti A. Numerical Simulation of Compressible Two–Phase Flow Model Based on the Riemann Problem. *ACOMEN 2002*, May 28–31, 2002, Liège University, Belgium.

[9] Zeidan D., Toro E. F. and Slaouti A. On the Riemann Problem for a Hyperbolic Two–Phase Flow Model. $2^{nd}$ *European Multiphase Systems Institute Meeting and* $40^{th}$ *European Two–Phase Flow Group Meeting*, June 10–13, 2002, Royal Institute of Technology, Stockholm, Sweden.

[10] Zeidan D. *Mathematical and Numerical Study of Two–Phase Flow Models*, PhD Thesis, Manchester Metropolitan University, Manchester, UK, 2003.