A NEW FINITE DIFFERENCE SCHEME BASED ON CENTRAL DIFFERENCE APPROXIMATION ASSOCIATED WITH HERONIAN MEAN AVERAGING FOR THE GOURSAT PROBLEM

MOHD. AGOS SALIM BIN NASIR
Department of Engineering Sciences,
Universiti Teknologi MARA,
Penang Branch Campus, 13500 Permatang Pauh,
Malaysia.
Tel: +604-3823428, Fax: +604-3822776
mohdagossalimnasir@yahoo.com

AHMAD IZANI MD. ISMAIL
School of Mathematical Sciences,
Universiti Sains Malaysia,
11800 Minden, Pulau Pinang,
Malaysia.
Tel: +604-6585508
izani@cs.usm.my

***Abstract*: -** The Goursat partial differential equation is a hyperbolic partial differential equation which arises in various field of study. Several approaches have been suggested using forward difference for developing schemes which approximate the Goursat partial differential equation. However it is known that the central difference approximation is more accurate than forward difference. In this paper we develop a new finite difference scheme for the Goursat partial differential equation using central differences together with Heronian mean averaging of functional values.

Keywords: Goursat problem, Finite difference schemes, Heronian mean, Central difference, Forward difference

1 INTRODUCTION

The Goursat problem arises in many areas of scientific applications, such as applied physics, environmental sciences, engineering etc. Researchers such as [7] and [2] have studied the problem theoretically while applications were studied by [6], [3], [10], [1], [8] and [11]. Finite difference schemes have been widely used to solve partial differential equations. These schemes involve the replacement of derivatives in the equations by the corresponding forward, backward or central difference approximations. Several finite difference schemes combined with various means of functional values have been developed for the Goursat problem ([5]; [13]).

In this paper we develop a new finite difference scheme for the Goursat problem based on central differences and apply this scheme to two Goursat problems.

2 THE GOURSAT PROBLEM, FINITE DIFFERENCE SCHEMES AND HERONIAN MEAN

The Goursat problem is of the form [13]:

$$\begin{aligned}&u_{xy} = f(x, y, u, u_x, u_y)\\&u(x,0) = \phi(x), u(0, y) = \psi(y), \phi(0) = \psi(0)\\&0 \le x \le a, 0 \le y \le b\end{aligned} \quad \ldots(1)$$

The left hand side of equation (1) can be discretized by using forward difference approximations as follows:

$$\begin{aligned}\frac{\partial^2 u}{\partial x \partial y} &= \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)\\&= \frac{\partial}{\partial y}\left(\frac{u(x+\Delta x, y) - u(x, y)}{\Delta x}\right)\\&= \frac{1}{\Delta x}\frac{\partial}{\partial y}\big(u(x+\Delta x, y) - u(x, y)\big)\\&= \frac{1}{\Delta x}\left[\frac{(u(x+\Delta x, y+\Delta y) - u(x+\Delta x, y))}{\Delta y} - \frac{(u(x, y+\Delta y) - u(x, y))}{\Delta y}\right]\\&= \frac{1}{h^2}\big[u(x+\Delta x, y+\Delta y) - u(x+\Delta x, y)) - u(x, y+\Delta y) + u(x, y)\big]\end{aligned}$$

where

$h = \Delta x = \Delta y$

…(2)

By indexing the variables, equation (2) becomes:

$$u_{xy} = \frac{u_{i+1,j+1} + u_{i,j} - u_{i+1,j} - u_{i,j+1}}{h^2} \quad \ldots(3)$$

If central difference approximations are used, we obtain:

$$\begin{aligned}\frac{\partial^2 u}{\partial x \partial y} &= \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)\\&= \frac{\partial}{\partial y}\left(\frac{u(x+\Delta x, y) - u(x-\Delta x, y)}{2\Delta x}\right)\\&= \frac{1}{2\Delta x}\frac{\partial}{\partial y}\big(u(x+\Delta x, y) - u(x-\Delta x, y)\big)\\&= \frac{1}{2\Delta x}\left[\begin{matrix}\dfrac{(u(x+\Delta x, y+\Delta y) - u(x+\Delta x, y-\Delta y))}{2\Delta y} - \\ \dfrac{(u(x-\Delta x, y+\Delta y) - u(x-\Delta x, y-\Delta y))}{2\Delta y}\end{matrix}\right]\\&= \frac{1}{4h^2}\left[\begin{matrix}u(x+\Delta x, y+\Delta y) - u(x+\Delta x, y-\Delta y) - \\ u(x-\Delta x, y+\Delta y) + u(x-\Delta x, y-\Delta y)\end{matrix}\right]\end{aligned}$$

where $h = \Delta x = \Delta y$

…(4)

Thus finite difference approximation for u_{xy} is:

$$u_{xy} = \frac{u_{i+1,j+1} - u_{i+1,j-1} - u_{i-1,j+1} + u_{i-1,j-1}}{4h^2} \quad \ldots(5)$$

Let x and y be positive numbers. The heronian mean of order 1 ("the heronian mean") denoted by H(x,y) is defined as [12]:

$$H(x, y) = \frac{x + \sqrt{xy} + y}{3} \quad \ldots(6)$$

The relationships between the arithmetic, geometric and heronian mean is given by H = (2A + G)/3 where A is the arithmetic mean (x+y)/2 and G is the geometric mean ($\sqrt{xy}$).

The R.H.S of equation (1) is approximated at (i + 1/2, j + 1/2) as follows:

$$f_{i+\frac{1}{2},j+\frac{1}{2}} = \text{Heronian mean (Heronian mean } f_{i+1,j} \text{ and } f_{i,j+1}\text{ ; Heronian mean } f_{i+1,j+1} \text{ and } f_{i,j})$$

…(7)

From (7) we obtain:

$$f_{i+\frac{1}{2},j+\frac{1}{2}} = \frac{1}{9}\left[\begin{array}{l}(f_{i+1,j+1}+\sqrt{f_{i+1,j+1}f_{i,j}}+f_{i,j})+\\ \sqrt{(f_{i+1,j+1}+\sqrt{f_{i+1,j+1}f_{i,j}}+f_{i,j})(f_{i+1,j}+\sqrt{f_{i+1,j}f_{i,j+1}}+f_{i,j+1})}+\\ (f_{i+1,j}+\sqrt{f_{i+1,j}f_{i,j+1}}+f_{i,j+1})\end{array}\right]$$

…(8)

Hence the scheme based on forward difference approximation as follows:

$$\frac{u_{i+1,j+1}+u_{i,j}-u_{i+1,j}-u_{i,j+1}}{h^2} = \frac{1}{9}\left[\begin{array}{l}(f_{i+1,j+1}+\sqrt{f_{i+1,j+1}f_{i,j}}+f_{i,j})+\\ \sqrt{(f_{i+1,j+1}+\sqrt{f_{i+1,j+1}f_{i,j}}+f_{i,j})(f_{i+1,j}+\sqrt{f_{i+1,j}f_{i,j+1}}+f_{i,j+1})}+\\ (f_{i+1,j}+\sqrt{f_{i+1,j}f_{i,j+1}}+f_{i,j+1})\end{array}\right]$$

…(9)

and the new scheme based on central difference approximation as follows:

$$\frac{u_{i+1,j+1}-u_{i+1,j-1}-u_{i-1,j+1}+u_{i-1,j-1}}{4h^2} = \frac{1}{9}\left[\begin{array}{l}(f_{i+1,j+1}+\sqrt{f_{i+1,j+1}f_{i,j}}+f_{i,j})+\\ \sqrt{(f_{i+1,j+1}+\sqrt{f_{i+1,j+1}f_{i,j}}+f_{i,j})(f_{i+1,j}+\sqrt{f_{i+1,j}f_{i,j+1}}+f_{i,j+1})}+\\ (f_{i+1,j}+\sqrt{f_{i+1,j}f_{i,j+1}}+f_{i,j+1})\end{array}\right]$$

…(10)

Henceforth, we shall refer to the finite difference scheme (9) as the forward difference scheme and the new scheme (10) as the central difference scheme.

We note that this implementation of the new central difference scheme (10) requires that u values on the first grid line in the x and y directions be known. This can be computed using the scheme (9).

3 NUMERICAL EXPERIMENTS

We consider the non-linear Goursat problem:

$$\begin{aligned}&u_{xy} = e^{2u}\\&u(x,0) = \frac{x}{2} - \ln(1+e^x)\\&u(0,y) = \frac{y}{2} - \ln(1+e^y)\\&0 \le x \le 4,\quad 0 \le y \le 4\end{aligned}$$

…(11)

The analytical solution of problem (11) is [13]:

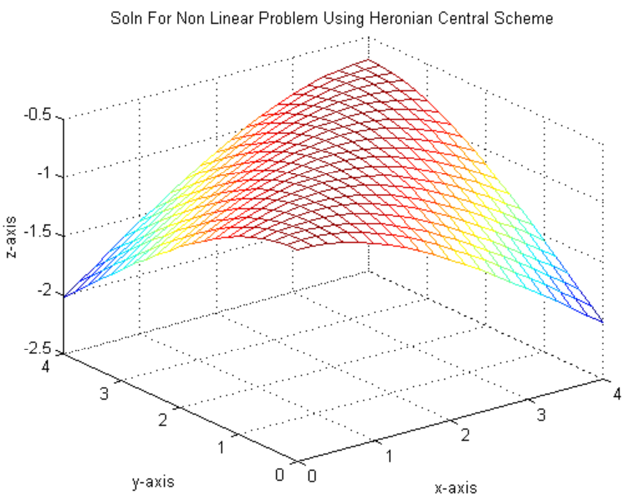
$$u(x,y)=\frac{x+y}{2}-\ln(e^{x}+e^{y})$$

Results of Numerical Experiments:

For grid sizes h = 0.025, 0.05, 0.1 and 0.2 we obtained the following results:

The Time Steps	Average error of Heronian central scheme	Average error of Heronian forward scheme
h = 0.025	1.5176833e-005	7.7545350e-005
h = 0.05	5.7753022e-005	3.1465180e-004
h = 0.1	2.0528428e-004	1.2941443e-003
h = 0.2	5.8135113e-004	5.4554515e-003

Fig.1: Solution in graphic form with h = 0.05



We consider the derivative linear Goursat problem:

$$u_{xy}=\frac{u_x+u_y+u}{3}$$

$$u(x,0)=e^{x}$$

$$u(0,y)=e^{y}$$

$$0\le x\le 4,\ 0\le y\le 4 \qquad \ldots(12)$$

The analytical solution for the Goursat problem (12) is [4]:

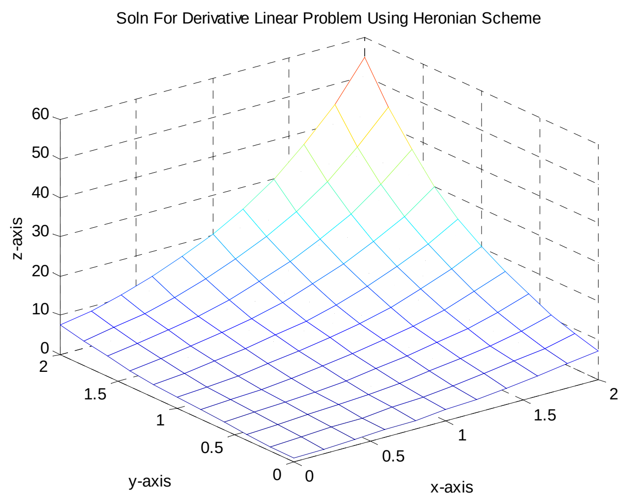
$$u(x,y)=e^{x+y}$$

Results of Numerical Experiments:

For grid sizes h = 0.005, 0.010, 0.020 and 0.025 we obtained the following results:

The Time Steps	Average error of Heronian central scheme	Average error of Heronian forward scheme
h = 0.005	8.5210105e-004	8.5530845e-004
h = 0.010	1.6988128e-003	1.7115304e-003
h = 0.020	3.3766949e-003	3.4266802e-003
h = 0.025	4.2081718e-003	4.2855907e-003

Fig.2: Solution in graphic form with h = 0.010



It can be seen that the Heronian central scheme gives better accuracy for the non linear and the derivative linear Goursat problem.

4 CONCLUSIONS

In this paper we have developed a new finite difference scheme for the Goursat problem. This scheme which uses central difference approximation (with heronian mean averaging of functional values) is more accurate than the forward difference scheme for the two Goursat problems considered. However the use of the forward difference scheme is required to compute some starting values.

References:

[1] Y.L-An, W.C.-Hua, The discontinuous ivp of a reacting gas flow system. Transactions of the American Mathematical Society 1981.

[2] A.K. Aziz & B.S. Hubbard, Bounds on the truncation error by finite differential for the Goursat problem. Mathematics of Computation, 1964.

[3] T.Y. Cheung, Three non-linear ivp's of the hyperbolic type. SIAM Journal of Numerical Analysis, 1978.

[4] J.T. Day A Runge-Kutta method for the numerical solution of the Goursat problem in hyperbolic partial differential equations.

[5] D.J. Evans & B.B. Sanugi, Numerical solution of the Goursat problem by a nonlinear trapezoidal formula. Applied Mathematics Letter, 1988.

[6] R.A. Frish & B.R. Chao, On a bounded one-dimensional Poisson-Vlasov system, SIAM Journal of Applied Mathematics, 1973.

[7] P.R. Garabedian, Partial differential equations. John Wiley & Sons Inc., 1964.

[8] P. Hillion, A note on the derivation of paraxial equation in non homogeneous media. SIAM Journal of Applied Mathematics, 1992.

[9] A.I.M. Ismail, M.A.S. Nasir & A.Z. Talib, Numerical Solution of the Goursat problem. Proceedings of the IASTED International Conference On Applied Simulation And Modeling, Rhodes Greece, June 2004.

[10] D.J. Kaup & A.C. Newell, The Goursat and Cauchy problems for the Sine Gordon equation. SIAM Journal of Applied Mathematics, 1978.

[11] J.R. Mclaughlin, P.L. Polyakov, P.E. Sacks, Reconstruction of a spherical symmetric speed of sound. SIAM Journal of Applied Mathematics, 1994.

[12] E. Neuman, A Generalization of An Inequality of JIA and CAU. Journal of Inequalities in Pure and Applied Mathematics, Issue 1, Vol. 5, ISSN (electronic): 1443-5756, February 2004.

[13] A.M. Wazwaz, On the numerical solution for the Goursat problems. Applied Mathematics and Computation, 1993.

[14] A.M. Wazwaz, The decomposition method for
approximate solution of the Goursat problem.
Applied Mathematics and Computation, 1995.