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Numerical Simulation for Fuzzy Fredholm Integral Equations Using Reproducing Kernel Algorithm

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Abstract- **In this paper, we simulate the numerical solutions of fuzzy Fredholm integral equations based on the reproducing kernel algorithm. Using parametric form of fuzzy numbers we convert a linear fuzzy integral equation into a linear system of integral equations in crisp case. The solution methodology is based on generating the orthogonal basis from the obtained kernel functions; whilst the orthonormal basis is constructing in order to formulate and utilize the solutions with series form in terms of their parametric form in an appropriate space. Numerical example is provided to illustrate potentiality of our algorithm for solving such fuzzy equations.**

Keywords- Fuzzy Fredholm integral equations; Reproducing kernel algorithm

I. INTRODUCTION

The fuzzy Fredholm integral equations (FFIEs) are important part of the fuzzy analysis theory and they have the important value of theory and application in control theory, measure theory, and radiation transfer in a semi-infinite atmosphere. Generally, many real-world problems are too complex to be defined in precise terms; uncertainty is often involved in any real-world design process. Fuzzy sets provide a widely appreciated tool to introduce uncertain parameters into mathematical applications. In many applications, at least some of the parameters of the model should be represented by fuzzy numbers rather than crisp numbers. Thus, it is immensely important to develop appropriate and applicable algorithm to accomplish the mathematical construction that would appropriately treat FFIEs and solve them.

The aim of this paper is to extend the application of the reproducing kernel Hilbert space (RKHS) method to provide numerical solution for the linear FFIEs of the form

$$x(t) = \int_0^1 h(t,\tau)x(\tau)d\tau, 0 \leq \tau, t \leq 1, \qquad (1)$$

where $h(t,\tau)$ is continuous arbitrary crisp kernel functions over the square $0 \leq \tau, t \leq 1$. Here, $\mathbb{R}_{\mathcal{F}}$ denote the set of fuzzy numbers on $\mathbb{R}$.

Reproducing kernel theory has important application in numerical analysis, computational mathematics, image processing, machine learning, finance, and probability and statistics [1-4]. Recently, a lot of research work has been devoted to the applications of the RKHS method for wide classes of stochastic and deterministic problems involving operator equations, differential equations, integral equations, and integro-differential equations. The RKHS method was successfully used by many authors to investigate several scientific applications side by side with their theories. The reader is kindly requested to go through [5-13] in order to know more details about the RKHS method, including its history, its modification for use, its scientific applications, its symmetric kernel functions, and its characteristics.

The RKHS method possess several advantages; first, it is of global nature in terms of the solutions obtained as well as its ability to solve other mathematical, physical, and engineering problems; second, it is accurate and need less effort to achieve the results; third, in the RKHS method, it is possible to pick any point in the interval of integration and as well the approximate solution will be applicable; fourth, the method does not require discretization of the variables, and it is not effected by computation round off errors and one is not faced with necessity of large computer memory and time.

Recently, the numerical solvability of FFIEs has been studied by several authors using different numerical or analytical methods. The reader is asked to refer to [14-17] in order to know more details about these analyzes and methods, including their kinds and history, their modifications and conditions for use, their scientific applications, their importance and characteristics, and their relationship including the differences.

The organization of the paper is as follows. In the next section, we present some necessary definitions and preliminary results from the fuzzy calculus theory. The

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procedure of solving FFIEs is presented in section III. In section IV, reproducing kernel algorithm is built and introduced. Numerical algorithm and simulation results are presented in Section V. This article ends in section VI with some concluding remarks.

II. FUZZY CALCULUS THEORY

Fuzzy calculus is the study of theory and applications of integrals and derivatives of uncertain functions. This branch of mathematical analysis, extensively investigated in the recent years, has emerged as an effective and powerful tool for the mathematical modeling of several engineering and scientific phenomena. In this section, we present some necessary definitions from fuzzy calculus theory and preliminary results.

Let X be a nonempty set, a fuzzy set u in X is characterized by its membership function $u: X \to [0,1]$. Thus, $u(s)$ is interpreted as the degree of membership of an element s in the fuzzy set u for each $s \in X$. A fuzzy set u on $\mathbb{R}$ is called convex, if for each $s, t \in \mathbb{R}$ and $\lambda \in [0,1]$, $u(\lambda s + (1-\lambda)t) \geq \min\{u(s), u(t)\}$, is called upper semicontinuous, if $\{s \in \mathbb{R}: u(s) > r\}$ is closed for each $r \in [0,1]$, and is called normal, if there is $s \in \mathbb{R}$ such that $u(s) = 1$. The support of a fuzzy set u is defined as $\{s \in \mathbb{R}: u(s) > 0\}$.

Definition II.1 [18] A fuzzy number u is a fuzzy subset of the real line with a normal, convex, and upper semicontinuous membership function of bounded support.

For each $r \in (0,1]$, set $[u]^r = \{s \in \mathbb{R}: u(s) \geq r\}$ and $[u]^0 = \overline{\{s \in \mathbb{R}: u(s) > 0\}}$. Then, it easily to establish that u is a fuzzy number if and only if $[u]^r$ is compact convex subset of $\mathbb{R}$ for each $r \in [0,1]$ and $[u]^1 \neq \phi$ [19]. Thus, if u is a fuzzy number, then $[u]^r = [u_1(r), u_2(r)]$, where $u_1(r) = \min\{s: s \in [u]^r\}$ and $u_2(r) = \max\{s: s \in [u]^r\}$ for each $r \in [0,1]$. The symbol $[u]^r$ is called the r-cut representation or parametric form of a fuzzy number u.

Theorem II.1 [19] Suppose that the functions $u_1, u_2: [0,1] \to \mathbb{R}$ satisfy the following conditions; first, u_1 is a bounded increasing and u_2 is a bounded decreasing with $u_1(1) \leq u_2(1)$; second, for each $k \in (0,1]$, u_1 and u_2 are left-hand continuous at $r = k$; third, u_1 and u_2 are right-hand continuous at $r = 0$. Then $u: \mathbb{R} \to [0,1]$ defined by $u(s) = \sup\{r: u_1(r) \leq s \leq u_2(r)\}$, is a fuzzy number with parameterization $[u_1(r), u_2(r)]$. Furthermore, if $u: \mathbb{R} \to [0,1]$ is a fuzzy number with parameterization $[u_1(r), u_2(r)]$, then the functions u_1 and u_2 satisfy the aforementioned conditions.

In general, we can represent an arbitrary fuzzy number u by an order pair of functions (u_1, u_2) which satisfy the requirements of Theorem II.1. Frequently, we will write simply u_{1r} and u_{2r} instead of $u_1(r)$ and $u_2(r)$, respectively.

The metric structure on $\mathbb{R}_{\mathcal{F}}$ is given by $d_\infty: \mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}} \to \mathbb{R}^+ \cup \{0\}$ such that $d_\infty(u,v) = \sup_{r\in[0,1]} \max\{|u_{1r} - v_{1r}|, |u_{2r} - v_{2r}|\}$ for arbitrary fuzzy numbers u and v. It is shown in [20] that $(\mathbb{R}_{\mathcal{F}}, d_\infty)$ is a complete metric space.

Definition II.2 [19] Suppose that $x: [0,1] \to \mathbb{R}_{\mathcal{F}}$, for each partition $\wp = \{t_0^*, t_1^*, \dots, t_n^*\}$ of $[0,1]$ and for arbitrary points $\xi_i \in [t_{i-1}^*, t_i^*]$, $1 \leq i \leq n$, let $\mathfrak{R}_\wp = \sum_{i=1}^n x(\xi_i)(t_i^* - t_{i-1}^*)$ and $\Delta = \max_{1\leq i\leq n} |t_i^* - t_{i-1}^*|$. Then the definite integral of $x(t)$ over $[0,1]$ is defined by $\int_0^1 x(t)dt = \lim_{\Delta\to 0} \mathfrak{R}_\wp$ provided the limit exists in the metric space $(\mathbb{R}_{\mathcal{F}}, d_\infty)$.

Theorem II.2 [19] Let $x: [0,1] \to \mathbb{R}_{\mathcal{F}}$ be continuous fuzzy-valued function and put $[x(t)]^r = [x_{1r}(t), x_{2r}(t)]$ for each $r \in [0,1]$. Then $\int_0^1 x(t)dt$ exist, belong to $\mathbb{R}_{\mathcal{F}}$, x_{1r} and x_{2r} are integrable functions on [0,1], and $\left[\int_0^1 x(t)dt\right]^r = \int_0^1 [x(t)]^r dt = \left[\int_0^1 x_{1r}(t)dt, \int_0^1 x_{2r}(t)dt\right]$.

III. SOLVING FFIEs

In this section, we study FFIEs using the concept of Riemann integrability in which the FFIE is converted into equivalent system of crisp integral equations (CIEs). Furthermore, an efficient computational algorithm is provided to guarantee the procedure.

Next, FFIE (1) is first formulated as an ordinary set of integral equations, after that, a new discretized form of FFIE (1) is presented. Anyhow, in order to apply our RKHS algorithm, we set $H(t,\tau,x(\tau)) := h(t,\tau)x(\tau)$, further, we write the fuzzy function $x(t)$ in term of its r-cut representation forms to get $[x(t)]^r = [x_{1r}(t), x_{2r}(t)]$. By considering the parametric form for both sides of FFIE (1), one can write

$$[x(t)]^r = \int_0^1 \left[H(t,\tau,x(\tau))\right]^r d\tau,$$

where $[H]^r = [H_{1r}, H_{2r}]$ in which H_{1r}, H_{1r} are given in the form of $H_{1r} = \min\{h(t,\tau)x_{1r}(\tau), h(t,\tau)x_{2r}(\tau)\}$ and $H_{2r} = \max\{h(t,\tau)x_{1r}(\tau), h(t,\tau)x_{2r}(\tau)\}$.

Prior to applying the RKHS methods for solving FFIE (1) in its parametric form, we suppose that the crisp kernel function $h(t,\tau)$ is nonnegative for $0 \leq \tau \leq c_1$ and nonpositive for $c_1 \leq \tau \leq 1$. Therefore, according to the previous results the FFIE (1) can be translated into the following equivalent form:

$$\begin{aligned} x_{1r}(t) &= \int_0^{c_1} h(t,\tau)x_{1r}(\tau)d\tau + \int_{c_1}^1 h(t,\tau)x_{2r}(\tau)d\tau, \\ x_{2r}(t) &= \int_0^{c_1} h(t,\tau)x_{2r}(\tau)d\tau + \int_{c_1}^1 h(t,\tau)x_{1r}(\tau)d\tau. \end{aligned} \quad (2)$$

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Definition III.1 Let $x:[0,1] \to \mathbb{R}_{\mathcal{F}}$ be continuous fuzzy-valued function. If x satisfy FFIE (1), then we say that x is a fuzzy solution of FFIE (1).

The object of the next algorithm is to implement a procedure to solve FFIE (1) in parametric form in term of its r-cut representation, where the new obtained system consists of two CIEs.

Algorithm III.1 To find the fuzzy solution of FFIE (1), we discuss the following main steps:

Input: The independent interval $[0,1]$, and the unit truth interval $[0,1]$.

Output: The fuzzy solution of FFIE (1) on $[0,1]$.

Step 1: Set $[H]^r = [H_{1r}, H_{2r}]$,

Step 2: Solve the following system of CIEs for $x_{1r}(t)$ and $x_{2r}(t)$:

$$x_{1r}(t) = \int_0^{c_1} h(t,\tau)x_{1r}(\tau)d\tau + \int_{c_1}^1 h(t,\tau)x_{2r}(\tau)d\tau,$$

$$x_{2r}(t) = \int_0^{c_1} h(t,\tau)x_{2r}(\tau)d\tau + \int_{c_1}^1 h(t,\tau)x_{1r}(\tau)d\tau.$$

Step 3: Ensure that the solution $[x_{1r}(t), x_{2r}(t)]$ are valid level sets for each $r \in [0,1]$.

Step 4: Construct the fuzzy solution $x(t)$ such that $[x(t)]^r = [x_{1r}(t), x_{2r}(t)]$ for each $r \in [0,1]$.

Step 5: Stop.

IV. REPRODUCING KERNEL ALGORITHM

In this section, we utilize the reproducing kernel concept in order to construct the reproducing kernel Hilbert space $W_2^m[0,1]$.

Prior to discussing the applicability of the RKHS method on solving FFIEs and their associated numerical algorithms, it is necessary to present an appropriate brief introduction to preliminary topics from the reproducing kernel theory.

Definition IV.1 [4] Let $\mathcal{H}$ be a Hilbert space of function $\phi:\Omega \longrightarrow \mathcal{F}$ on a set Ω. A function $K:\Omega\times\Omega \longrightarrow \mathbb{C}$ is a reproducing kernel of $\mathcal{H}$ if the following conditions are satisfied. Firstly, $K(.,t) \in \mathcal{H}$ for each $t \in \Omega$. Secondly, $\langle \phi, K(.,t)\rangle = \phi(t)$ for each $\phi \in \mathcal{H}$ and each $t \in \Omega$.

The second condition in Definition IV.1 is called "the reproducing property" which means that, the value of the function ϕ at the point t is reproduced by the inner product of ϕ with $K(.,t)$. Indeed, a Hilbert spaces $\mathcal{H}$ of functions on a nonempty abstract set Ω is called a reproducing kernel Hilbert spaces if there exists a reproducing kernel K of $\mathcal{H}$.

Definition IV.2 The inner product space $W_2^m[0,1]$ is defined as $W_2^m[0,1] = \{z: z, z', \ldots, z^{(m-1)}$ are absolutely continuous real-valued function on $[0,1]$ and $z^{(m)} \in L^2[0,1]\}$. The inner product and the norm in $W_2^m[0,1]$ are defined as $\langle z_1(t), z_2(t)\rangle_{W_2^m} = \sum_{i=0}^{m-1} z_1^{(i)}(0)z_2^{(i)}(0) + \int_0^1 z_1^{(m)}(t)z_2^{(m)}(t)dt$ and $\|z_1\|_{W_2^m} = \sqrt{\langle z_1(t), z_1(t)\rangle_{W_2^m}}$, respectively, where $z_1, z_2 \in W_2^m[0,1]$.

The Hilbert space $W_2^m[0,1]$ is called a reproducing kernel if for each fixed $t \in [0,1]$ and any $z(s) \in W_2^m[0,1]$, there exist $K(t,s) \in W_2^m[0,1]$ (simply $K_t(s)$) and $s \in [0,1]$ such that $\langle z(s), K_t(s)\rangle_{W_2^m} = z(t)$.

Theorem IV.1 The Hilbert space $W_2^m[0,1]$ is a complete reproducing kernel and its reproducing kernel function $R_t^m(s)$ can be written as

$$R_t^m(s)|_{s\le t} = \sum_{i=0}^{m-1} \frac{1}{(i!)^2} t^i s^i + \frac{1}{\left((m-1)!\right)^2} \int_0^s (t-\tau)^{m-1}(s-\tau)^{m-1} d\tau$$

$$R_t^m(s)|_{s>t} = \sum_{i=0}^{m-1} \frac{1}{(i!)^2} t^i s^i + \frac{1}{\left((m-1)!\right)^2} \int_0^t (t-\tau)^{m-1}(s-\tau)^{m-1} d\tau$$

Definition IV.3 The inner product space $W^m[0,1]$ is defined as $W^m[0,1] = \{z = (z_1, z_2)^T : z_1, z_2 \in W_2^m[0,1]\}$. The inner product and the norm in $W^m[0,1]$ are building as $\langle z(t), w(t)\rangle_{W^m} = \sum_{i=1}^2 \langle z_i(t), w_i(t)\rangle_{W_2^m}$ and $\|z\|_{W^m} = \sqrt{\sum_{i=1}^2 \|z_i\|_{W_2^m}^2}$, respectively, where $z(t), w(t) \in W^m[0,1]$.

To deal with System (2) in more realistic form via the RKHS approach, define the linear operator $v_{ij}: W_2^2[0,1] \to W_2^1[0,1]$, $i,j = 1,2$ such that

$$v_{ij}z(t) = \begin{cases} z(t) - \int_0^{c_1} h(t,\tau)z(\tau)d\tau, & i = j, \\ -\int_{c_1}^1 h(t,\tau)z(\tau)d\tau, & i \ne j. \end{cases}$$

Put $0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $X_r = \begin{bmatrix} x_{1r} \\ x_{2r} \end{bmatrix}$, $V = \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix}$, and define the mapping $V: W^2[0,1] \to W^1[0,1]$. Then, System (2) can be written in a new form equivalent to $VX_r(t) = 0$.

V. SIMULATION RESULT

To show behavior, properties, and applicability of the present RKHS method, linear FFIEs will be solved numerically in this section. In the process of computation, all the symbolic and numerical computations are performed by using MAPLE 13 software package.

Algorithm V.1 To approximate the solution $x^n(t)$ of $x(t)$ for FFIE (1), we do the following main steps:

Input: The dependent interval $[0,1]$, the unit truth interval $[0,1]$, the integers n, m, the kernel functions $R_t^1(s), R_t^2(s)$, the linear operator V, and the crisp kernel functions $h(t,\tau)$.

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Output: The RKHS solution $X_r^n(t)$ of $X_r(t)$ for System (2) and thus the RKHS solution $x^n(t)$ of $x(t)$ for FFIE (1).

Step 1: Write $X_r = \begin{bmatrix} x_{1r} \\ x_{2r} \end{bmatrix}$ and $X_r^n(t) = \begin{bmatrix} x_{1r}^n \\ x_{2r}^n \end{bmatrix}$.

Step 2: Fixed t in [0,1] and set $s \in [0,1]$;

If $s \le t$, set $R_t^2(s) = 1 + ts - \frac{1}{6}t^2(t - 3s)$;

Else set $R_t^2(s) = 1 + ts - \frac{1}{6}s^2(s - 3t)$;

For $i = 1,2,\dots,n, h = 1,2,\dots,m, j = 1,2$:

Set $t_i = \frac{i-1}{n-1}$;

Set $r_h = \frac{h-1}{m-1}$;

Set $\psi_{ij}(t) = \begin{bmatrix} [v_{j1}R_t^2(s)]\big|_{s=t_i} \\ [v_{j2}R_t^2(s)]\big|_{s=t_i} \end{bmatrix}$;

Output: the orthogonal function system $\psi_{ij}(t)$.

Step 3: For $l = 2,3\dots,n, k = 1,2\dots,l$:

Set $d_l = \sqrt{\|\psi_l\|_{W^2}^2 - \sum_{p=1}^{l-1} c_{lp}^2}$;

Set $c_{lk} = \langle \psi_l, \bar{\psi}_k \rangle_{W^2}$;

If $k \ne l$, then set $\beta_{lk} = -\frac{1}{d_l}\sum_{p=k}^{i-1} c_{lp}\beta_{pk}$;

Else set $\beta_{ll} = \frac{1}{d_l}$;

Else set $\beta_{11} = \frac{1}{\|\psi_1\|_{W^2}}$;

Output: the orthogonalization coefficients β_{lk}.

Step 4: For $l = 2,3\dots,n-1, k = 1,2\dots,l-1$:

Set $\bar{\mu}_l(t) = \sum_{k=1}^{l} \beta_{lk}\mu_k(t)$;

Output: the orthonormal function system $\bar{\mu}_l(t)$.

Step 5: Set $X_{r_h}^0(t_1) = 0$;

For $i = 1,2,\dots,n$:

Set $\alpha_k = \begin{cases} x_{1r}\left(t_{\frac{k+1}{2}}\right), & k \text{ is odd}, \\ x_{2r}\left(t_{\frac{k}{2}}\right), & k \text{ is even}; \end{cases}$

Set $X_{r_h}^i(t_i) = \sum_{i=1}^{i}\left(\sum_{k=1}^{i}\beta_{ik}\alpha_k\right)\bar{\mu}_i(t)$;

Output: the RKHS solution $X_r^n(t)$ of $X_r(t)$.

Step 6: Write $[x^n(t)]^r = [x_{1r}^n, x_{2r}^n]$ to get the RKHS solution in which $[x(t)]^r = [x_{1r}, x_{2r}]$.

Step 7: Stop.

Here, we taking $t_i = \frac{i-1}{n-1}$, $i = 1,2,\dots,n$ and $r_h = \frac{h-1}{m-1}$, $h = 1,2,\dots,m$ with the reproducing kernel functions $R_t^1(s)$ and $R_t^2(s)$ on $[0,1]$ in which Algorithms III.1 and V.1 are used throughout the computations.

Example V.1 Consider the following FFIE:

$$x(t) = \frac{1}{3}\pi\sin(\pi t)\,u + \int_0^1 \pi\sin(2\pi\tau)\sin(\pi t)\,x(\tau)d\tau,$$

where $0 \le \tau, t \le 1$. The exact solution is $x(t) = v\pi\sin(\pi t)$. Here, $[u]^r = [-5r^3 - 2r^2 - 7r + 20, 2r^3 + 5r^2 + 7r - 8]$ and $[v]^r = [-r^3 - r + 4, r^2 + r]$.

Anyhow, for approximating the fuzzy solution, we have the following system of CIEs taking into account that the crisp kernel function $h(t,\tau) = \pi\sin(2\pi\tau)\sin(\pi t)$ is nonnegative on $0 \le \tau \le \frac{1}{2}$ and nonpositive on $\frac{1}{2} \le \tau \le 1$, regardless the effect of the independent variable t on [0,1]:

$$\begin{aligned} x_{1r}(t) &= \frac{1}{3}\pi\sin(\pi t)\,u_{1r} \\ &\quad + \int_0^{\frac{1}{2}} \pi\sin(2\pi\tau)\sin(\pi t)\,x_{1r}(\tau)d\tau \\ &\quad + \int_{\frac{1}{2}}^1 \pi\sin(2\pi\tau)\sin(\pi t)\,x_{2r}(\tau)d\tau, \\ x_{2r}(t) &= \frac{1}{3}\pi\sin(\pi t)\,u_{2r} \\ &\quad + \int_0^{\frac{1}{2}} \pi\sin(2\pi\tau)\sin(\pi t)\,x_{2r}(\tau)d\tau \\ &\quad + \int_{\frac{1}{2}}^1 \pi\sin(2\pi\tau)\sin(\pi t)\,x_{1r}(\tau)d\tau. \end{aligned}$$

The absolute errors of numerically approximating $x_{1r}(t)$ and $x_{2r}(t)$ for the corresponding CIE system have been calculated for various t and r as shown in Tables 1 and 2. Anyhow, it is clear from the tables that, the approximate solutions are in close agreement with the exact solutions.

VI. CONCLUSION

The study of FFIEs forms a suitable setting for the mathematical modeling of real-world problems in which uncertainty or vagueness pervades. The aim of this paper is to propose a numerical method and the corresponding algorithm to solve linear FFIEs. Numerical results show that the presented method is of higher precision and is easy to apply in programming.

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Table 1: The absolute errors of approximating $x_{1r}(t)$ using RKHS method.

t_i	$r=0$	$r=0.25$	$r=0.5$	$r=0.75$	$r=1$
0.16	2.2167×10^{-8}	2.0686×10^{-8}	2.7409×10^{-8}	2.7281×10^{-8}	2.3836×10^{-8}
0.32	3.6612×10^{-8}	4.9143×10^{-8}	4.7620×10^{-8}	3.2652×10^{-8}	3.6097×10^{-8}
0.64	7.3751×10^{-8}	6.6190×10^{-8}	6.3982×10^{-8}	6.7446×10^{-8}	6.5208×10^{-8}
0.96	8.3262×10^{-8}	7.9022×10^{-8}	8.1784×10^{-8}	7.2881×10^{-8}	8.0809×10^{-8}

Table 2: The absolute errors of approximating $x_{2r}(t)$ using RKHS method.

t_i	$r=0$	$r=0.25$	$r=0.5$	$r=0.75$	$r=1$
0.16	2.3445×10^{-8}	2.1964×10^{-8}	2.8124×10^{-8}	2.3027×10^{-8}	2.6190×10^{-8}
0.32	3.7027×10^{-8}	3.7379×10^{-8}	3.5536×10^{-8}	3.3262×10^{-8}	3.4584×10^{-8}
0.64	6.8755×10^{-8}	6.9376×10^{-8}	6.2167×10^{-8}	6.7450×10^{-8}	6.4224×10^{-8}
0.96	7.6651×10^{-8}	8.8084×10^{-8}	7.5751×10^{-8}	8.6631×10^{-8}	8.1507×10^{-8}

REFERENCES

[1] H.L. Weinert, "Reproducing Kernel Hilbert Spaces: Applications in Statistical Signal Processing", Hutchinson Ross, 1982.

[2] A. Daniel, "Reproducing Kernel Spaces and Applications", Springer, Basel, Switzerland, 2003.

[3] A. Berlinet, C.T. Agnan, "Reproducing Kernel Hilbert Space in Probability and Statistics", Kluwer Academic Publishers, Boston, Mass, USA, 2004.

[4] M. Cui, Y. Lin, "Nonlinear Numerical Analysis in the Reproducing Kernel Space", Nova Science, New York, NY, USA, 2009.

[5] O. Abu Arqub, "An iterative method for solving fourth-order boundary value problems of mixed type integro-differential equations", Journal of Computational Analysis and Applications, In press.

[6] O. Abu Arqub, M. Al-Smadi, "Numerical algorithm for solving two-point, second-order periodic boundary value problems for mixed integro-differential equations", Applied Mathematics and Computation 243, 2014, pp. 911-922.

[7] O. Abu Arqub, M. Al-Smadi, S. Momani, "Application of reproducing kernel method for solving nonlinear Fredholm-Volterra integro-differential equations", Abstract and Applied Analysis, Volume 2012, Article ID 839836, 16 pages, 2012. doi:10.1155/2012/839836.

[8] O. Abu Arqub, M. Al-Smadi, N. Shawagfeh, "Solving Fredholm integro-differential equations using reproducing kernel Hilbert space method", Applied Mathematics and Computation 219, 2013, pp. 8938-8948.

[9] O. Abu Arqub, S. Momani, S. Al-Mezel, M. Kutbi, "A novel iterative numerical algorithm for the solutions of systems of fuzzy initial value problems", Applied Mathematics & Information Sciences, In Press.

[10] M. Al-Smadi, O. Abu Arqub, A. El-Ajuo, "A numerical method for solving systems of first-order periodic boundary value problems", Journal of Applied Mathematics, Volume 2014, Article ID 135465, 10 pages, 2014. doi:10.1155/2014/135465.

[11] M. Al-Smadi, O. Abu Arqub, S. Momani, "A computational method for two-point boundary value problems of fourth-order mixed integro-differential equations", Mathematical Problems in Engineering, Volume 2013, Article ID 832074, 10 pages, 2012, doi:10.1155/2013/832074.

[12] S. Momani, O. Abu Arqub, T. Hayat, H. Al-Sulami, "A computational method for solving periodic boundary value problems for integro-differential equations of Fredholm-Voltera type", Applied Mathematics and Computation 240, 2014, pp. 229-239.

[13] N. Shawagfeh, O. Abu Arqub, S. Momani, "Analytical solution of nonlinear second-order periodic boundary value problem using reproducing kernel method", Journal of Computational Analysis and Applications 16, 2014, pp. 750-762.

[14] H.S. Goghary, M.S. Goghary, "Two computational methods for solving linear Fredholm fuzzy integral equation of the second kind", Applied Mathematics and Computation 182, 2006, pp. 791-796.

[15] E. Babolian, H.S. Goghary, S. Abbasbandy, "Numerical solution of linear Fredholm fuzzy integral equations of the second kind by Adomian method", Applied Mathematics and Computation 161, 2005, pp. 733-744.

[16] S. Abbasbandy, E. Babolian, M. Alavi, "Numerical method for solving linear Fredholm fuzzy integral equations of the second kind", Chaos, Solitons and Fractals 31, 2007, pp. 138-146.

[17] A. Molabahrami, A. Shidfar, A. Ghyasi, "An analytical method for solving linear Fredholm fuzzy integral equations of the second kind", Computers and Mathematics with Applications 61, 2011, pp. 2754-2761.

[18] O. Kaleva, Fuzzy differential equations, "Fuzzy Sets and Systems" 24, 1987, pp. 301-317.

[19] R. Goetschel, W. Voxman, "Elementary fuzzy calculus", Fuzzy Sets and Systems 18, 1986, pp. 31-43.

[20] M.L. Puri, "Fuzzy random variables", Journal of Mathematical Analysis and Applications 114, 1986, pp. 409-422.